Influence of the Discrete Laplace and Fourier–Bessel Transforms on the Solution to Direct Seismic Problems within the Frequency Domain

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Abstract

Theoretical solution to the direct seismic problem for a thin layer elastic model of medium in the spectral domain can be constructed effectively using the Laplace and Fourier-Bessel transforms. However, for practical use, as a tool of modeling and the solution of inverse problems, it is important to analyze properties of the corresponding discrete transforms and their influence on the solution obtained. This research presents some of the results of our investigation of the properties of the transforms using functions giving analytically, which allows the possibility to analyze the most important features of the transforms. Such investigations permitted to determine the conditions for application of these transforms to the solution of considered problems in more details.

Introduction

Mitrofanov, Priimenko and Missagia (2007(a)) formulated an effective method for the solution of a direct problem for the Lamé system. The method uses the Laplace and Fourier-Bessel transforms to form an analog of the direct problem in the spectral domain. Such an approach gives the possibility to use the matrix Riccatti method for the solution of corresponding differential equations. The obtained solution has a high accuracy and a high performance. From the mathematical point of view the obtained solution is completed.

But in practical applications of the solution arise several serious problems which need a special investigation. One of them is the choice of the Laplace transform parameter. Another important problem is connected with using the solution of the direct problem in the optimization approach to the solution of the corresponding inverse problem into a frequency domain or a spectral domain. It has connection with the general problem of the combination of the theoretical solution with the observed data. In this case it is necessary to realize a transform of the observed multicomponent seismogram into the two-dimensional frequency domain using the Laplace and Fourier-Bessel transforms. Taking into account the properties of the discrete analogues of the transforms and the aperture limitation is very important.

It should be noted that in mathematical literature there are few publications about this subject.

Therefore, to talk about the discrete Laplace transform, it is possible to make some conclusions on the basis of well studied properties of the discrete Fourier transform, but in the case of the discrete Fourier-Bessel transform the question appears not to have been studied (for these types of problems).

Method

Study of numerical features of the considered transforms is done on the basis of functions given analytically. It has a number of advantages. Therefore, consideration of some properties of discrete analogues of the Laplace and Fourier-Bessel transforms is possible in the exact analytical form. In addition, it is possible to analyze accuracy of calculations for the constructed procedures in relation to analytical formulas.

As the base for the research the following function of two variables t and x was taken:

$$U(x,t) = \theta(t - t_0(x)) \cdot e^{-\alpha_0 (t - t_0(x))} \cdot e^{-\alpha_1 x} \cdot \cdot \cos(\omega_0 (t - t_0(x))), \qquad (1)$$

where $\theta(t - t_0(x))$ is usual Heaviside's function, which is equal to 1 for $t > t_0(x)$ and to 0 for $t < t_0(x)$. Thus, the function $U(x,t)$ can present a simple model of seismogram, $t_0(x)$ sets an initial time on the seismogram, and the parameters $\alpha_0, \alpha_1, \omega_0$ define the signal form.

The choice of the given function was defined by several features. Firstly, it allowed to connect the features of the transforms with the structure of observed seismic signals. Secondly, for this function it is possible to construct analytical expressions of the Laplace and Fourier-Bessel transforms. Thirdly, despite its elementary kind with its help it is possible to consider such important features, as influence of sharpness of the signal entry or change of arrival time of the signal on spatial coordinate x, on results of transformations.

Further we will analyze two cases of the function $t_0(x)$:

1. $t_0(x)$ does not depend from x and is the fixed constant.
2. $t_0(x)$ linearly depends from x.

It gives the opportunity to understand better the feature of discrete transforms with respect to the variation of the arrival time of the signal.

At construction of analytical expressions for corresponding transforms from the given function we will use the same formulas as have been used for construction of the solution of a direct problem in the frequency domain, see Mitrofanov, Priimenko and Missagia (2007(a)). The same formulas have been realized and by working out of procedures for calculation of two-dimensional discrete spectra from multicomponent seismogram for the solution of an inverse dynamic problem in spectral domain, see Mitrofanov, Priimenko and Missagia (2007(b)). In the general form they represent by the following integral:

$$\tilde{U}(\nu, f) = \int_0^\infty \int_0^\infty U(x,t) \cdot \mathrm{e}^{pt} x J_m(\nu x)\, dt dx, \qquad (2)$$

where $p = -\alpha + i\omega$ is the complex parameter of the Laplace transform, ν is the spatial frequency, $\omega = 2\pi f$ is the circular frequency, and f is measured in hertz. The index m indicates on the type of Bessel's function, and is equal to 0 for the vertical component of the observed wave field or to 1 for the horizontal component.

By substituting the integral (2) expression (1) it is possible to obtain formulas, as for the Laplace transform of separate traces, and for a two-dimensional spectrum, constructed using the Laplace and Fourier-Bessel transforms.

Results

For construction of analytical expressions using the Laplace transform it will be necessary to take advantage of fairly simple relation between two complex parameters p and s, where $s = \alpha + i\omega$. From here we have $p = -s^*$, where s^* means a complex conjugate with s value. Then, having allocated in expression (1) function:

$$\phi(x,\tilde{t}) = \theta(\tilde{t}) \cdot \mathrm{e}^{-\alpha_0 \tilde{t}} \cdot \cos(\omega_0 \tilde{t}), \qquad (3)$$

where $\tilde{t} = t - t_0(x)$, it is simple to show that:

$$\begin{aligned} \tilde{U}^{\Im}(x, f) &= \int_0^\infty \phi(x,\tilde{t}) \cdot \mathrm{e}^{p\tilde{t}} \cdot \mathrm{e}^{pt_0(x)}\, dt = \\ &= \mathrm{e}^{pt_0(x)} \int_0^\infty \phi(x,\tilde{t}) \cdot \mathrm{e}^{-s^* \tilde{t}}\, dt = \\ &= \mathrm{e}^{pt_0(x)} \cdot \frac{s^* + \alpha_0}{(s^* + \alpha_0)^2 + \omega_0^2} = \\ &= \mathrm{e}^{pt_0(x)} \cdot \left(\frac{s + \alpha_0}{(s + \alpha_0)^2 + \omega_0^2} \right)^*. \end{aligned} \qquad (4)$$

These allow us to define expressions for the Laplace transform of the analytically giving function (3), corresponding to the separate traces with fixed x. Using the obtained formulas (4), it is possible to define analytical formulas for a full two-dimensional spectrum of the given function $U(x,t)$:

$$\begin{aligned} \tilde{U}(\nu, f) &= \frac{-p + \alpha_0}{(-p + \alpha_0)^2 + \omega_0^2} \cdot \\ &\cdot \int_0^\infty \mathrm{e}^{pt_0(x)} \cdot \mathrm{e}^{-\alpha_1 x} x J_m(\nu x)\, dx. \end{aligned} \qquad (5)$$

It has been above specified that for the next analysis, formulas for $\tilde{U}(\nu, f)$, which correspond to two cases of the function $t_0(x)$ be required: I - $t_0(x) = t_1$, where t_1 is a fixed positive number, and II - $t_0(x) = t_1 x$. It is easy to show that in the first case expression (5) essentially becomes easier. In particular, for the seismograms, corresponded to the vertical component we obtain in the first case:

$$\begin{aligned} \tilde{U}(\nu, f) &= \frac{-p + \alpha_0}{(-p + \alpha_0)^2 + \omega_0^2} \cdot \\ &\cdot \frac{\alpha_1}{\left(\alpha_1^2 + \nu^2\right)^{3/2}} \cdot \mathrm{e}^{pt_1}. \end{aligned} \qquad (6)$$

In the second case construction of expressions for a two-dimensional spectrum possess greater complexity, but thus it is also possible to construct analytical formulas. So, for the vertical component we will have the following expression:

$$\begin{aligned} \tilde{U}(\nu, f) &= \frac{-p + \alpha_0}{(-p + \alpha_0)^2 + \omega_0^2} \cdot R^{-3} \cdot \\ &\cdot \left[(a_1 - i\omega_1)\cos(\frac{3}{2}\phi) - (\omega_1 + i a_1)\sin(\frac{3}{2}\phi) \right], \end{aligned} \qquad (7)$$

where $a_1 = \alpha_1 + \alpha t_1, \quad \omega_1 = \omega \cdot t_1$, and for R and φ we have the following representations:

$$R = \left[(a_1^2 + \nu^2 - \omega_1^2)^2 + 4\omega_1^2 a_1^2 \right]^{1/4},$$

$$\phi = \arg(a_1^2 + \nu^2 - \omega_1^2 - 2i\omega_1 a_1).$$

The formulas obtained were used further to carry out various test experiments. Some of these experiments are presented below.

Examples

For approach of the experiments to a seismic situation, values of theoretical function $U(x,t)$ have been presented in the form of 24 channel seismograms with distance on x between receivers to $50\,\mathrm{m}$ and carrying out of the first receiver on $100\,\mathrm{m}$ from the source position. The record time was $3\sec$. Thus, the aperture for test function was on the spatial coordinate $[100\,\mathrm{m}, 1250\,\mathrm{m}]$ and on the time coordinate $[0\sec, 3\sec]$. This kind of the data was used in all subsequent experiments.

Let's consider a series of the experiments with the Laplace transformation. These were done for the case I, when $t_1 = 0$ or $t_1>0$. The structure and parameters of the signals set by test function are given in Figure 1. For convenience further the function corresponding $\alpha_0 = 100$ with a short impulse (Figure 1 (a), we will call the first type of function, and the second function corresponding $\alpha_0 = 10$ with a long impulse, we will call the second type of function (Figure 1 (b)).

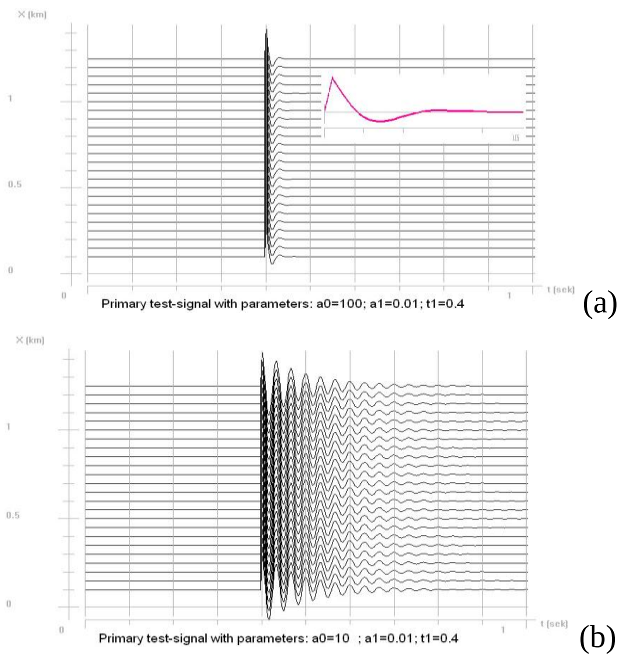

Figure 1

Figures 2-4 present results of some test experiments connected with the Laplace transform. Thus the structure of analytically calculated values of the real or imaginary component $\tilde{U}^{\Im}(x, f)$, which have been obtained with use of expressions (4), in all Figures is shown in the top part, i.e. corresponds to the letter (a). The results on the lower parts of the figure correspond to discrete analogue of this transform are shown.

The analysis of narrower frequency intervals show the influence of this component will be insignificant to frequencies less than $50\,\mathrm{Hz}$. At the same time, it is obvious that the solution of the inverse problem can need the frequencies exceeding $50\,\mathrm{Hz}$. Therefore, elimination of such trend components is required.

The results corresponding to Figures 2 and 3, represent behavior of an imaginary component of the function in a case for short (Figure 2) and a long impulse (Figure 3). It is visible that at both types of signals appears considerable trend component, essentially increasing in area of high frequencies (Figures 2 (b) and 3 (b), correspondingly).

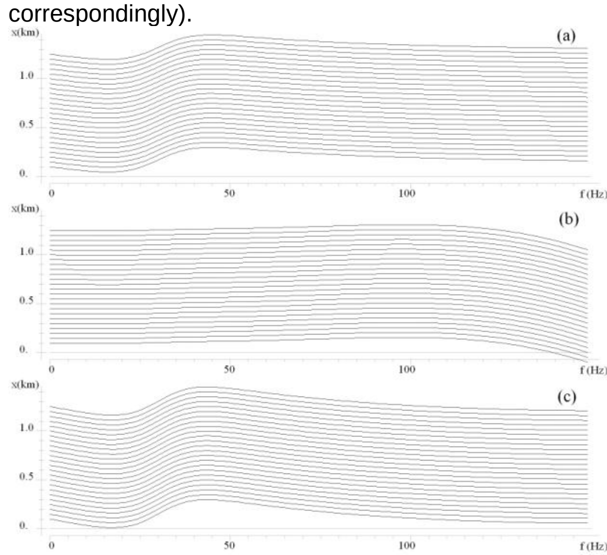

Figure 2

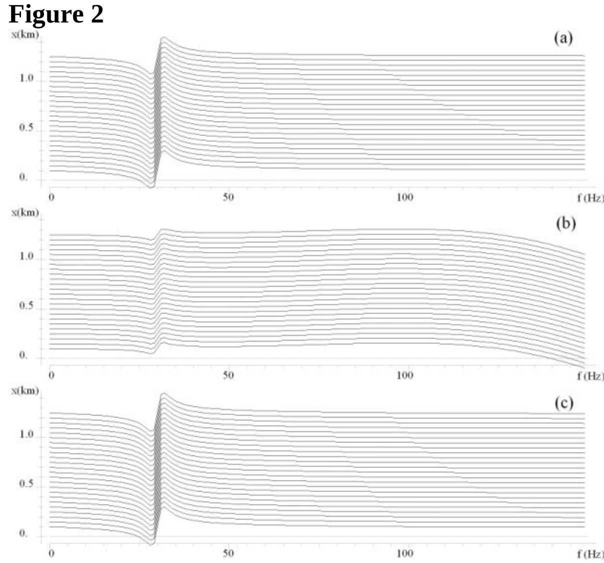

Figure 3

The experience with the discrete Fourier transform shows that similar trend components, considerably increasing on high frequencies, are connected with boundary effects of "cutting" of impulses from a seismic trace. Their elimination is simply enough provided with the use of special "windows", see Mitrofanov (1979). Application of similar "window" in the case of the discrete Laplace transform also provides substantial improvement to the quality of under construction spectra (Figures 2(c) and 3(c), correspondingly).

It is obvious that for the given structure of function in the case $t_1 > 0$ using special "windows" almost is not required. Therefore on the basis of the discrete Laplace transformation it is possible to calculate promptly values of spectra, which precisely correspond to the analytically calculated values of $\tilde{U}^{\Im}(x, f)$. On Figure 4 the real components of such spectra, calculated for a long impulse at $t_1 = 0{,}4\ \text{sec}$, are presented.

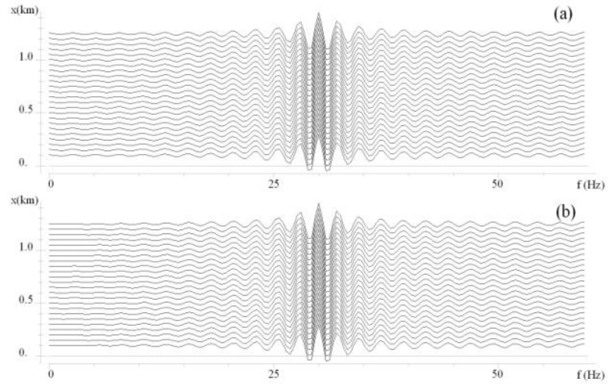

Figure 4

As a comment to the obtained results it is possible to specify that impulse expansion in time representation for $t_1 = 0$ leads to increase in compression of results of the Laplace transform. For $t_1 > 0$ in structure of real and imaginary components of $\tilde{U}^{\Im}(x, f)$ harmonic components appear. These results coincide with known properties of the Fourier transform.

Following on to the results which have been obtained using the two-dimensional spectra. As well as for the Laplace transform we will begin consideration with the case $t_1 = 0$. At representation of the obtained results we will use only the real component of the spectrum. For the imaginary component of the spectrum results were similar.

The results shown in Figure 5 demonstrate that transfer of the seismograms in the spectral domain by direct calculation of integral (2) will lead to essential distortions of the spectra. Except noted above a trend in the area of high time frequencies we will receive some pulsation in the spectrum at change of values of spatial frequency, see Figure 5 (b). Specified effects, both in case of the given experiment, and in case of other experiments, are eliminated by use of various types of "windows". Thus for elimination of a trend on high time frequencies "windows" were used similar to that, which were used for the Laplace transform, see Figure 5(c). For elimination of the pulsation smoother "windows" were used on the spatial variable, see Figure 5(d). The last were applied to the function $\tilde{U}^{\Im}(x, f)$, obtained after application of the discrete Laplace transform. Application of both types of "windows" has allowed to receive two-dimensional spectra which well enough coincided with analytical values, see Figure 5(f).

All obtained results testified that the constructed procedures of the discrete Laplace and Fourier-Bessel transforms for transfer of the seismograms in the frequency domain worked stably and allowed to obtain results close to analytical expressions. Therefore all subsequent experiments were already used for the purpose of finding-out of features of such conversions. In particular, it was analyzed influences of dependence of time of arrival of a signal from spatial coordinate on structure of defined two-dimensional spectra. Here it has been shown that the divergence between an analytical spectrum and a spectrum calculated by means of discrete transforms in this case can be considerable. It is connected with influence of the real aperture of observations.

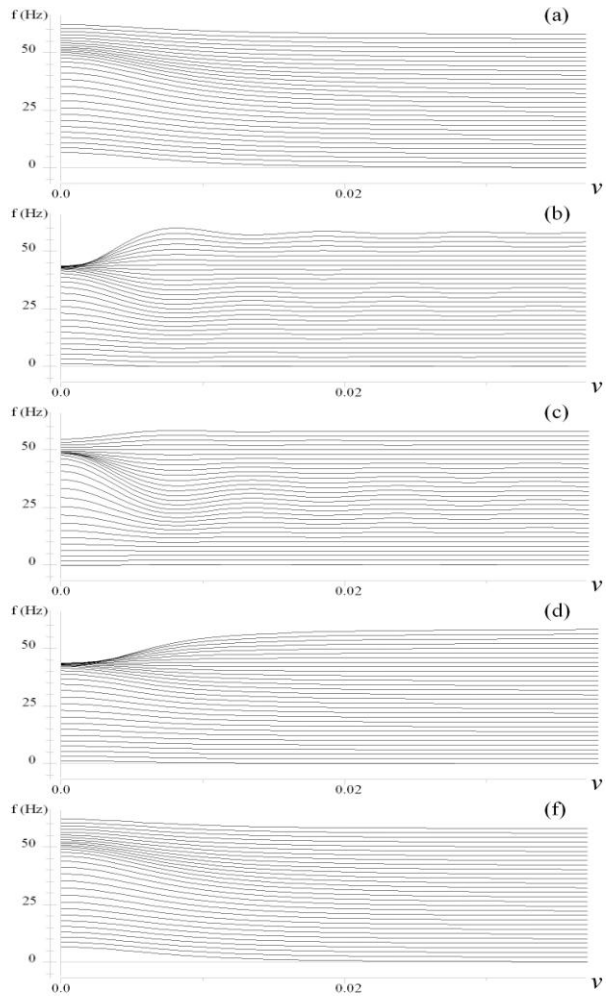

Figure 5

Consider the examples that show distinction between the calculated and analytical two-dimensional spectra, arising at change of arrival time of a signal. The sense of this distinction consists of the following. Integration in an infinite limit on the spatial variable in expression (2) allows to "feel" even small changes of the function $t_0(x)$ in structure of a two-dimensional spectrum $\tilde{U}(\nu, f)$. At the time limitation of the aperture and finiteness of digitization of seismograms on t and x does not give such possibility for a discrete spectrum. Thus, in the second case of the function $t_0(x)$ we start to face the specified problems directly.

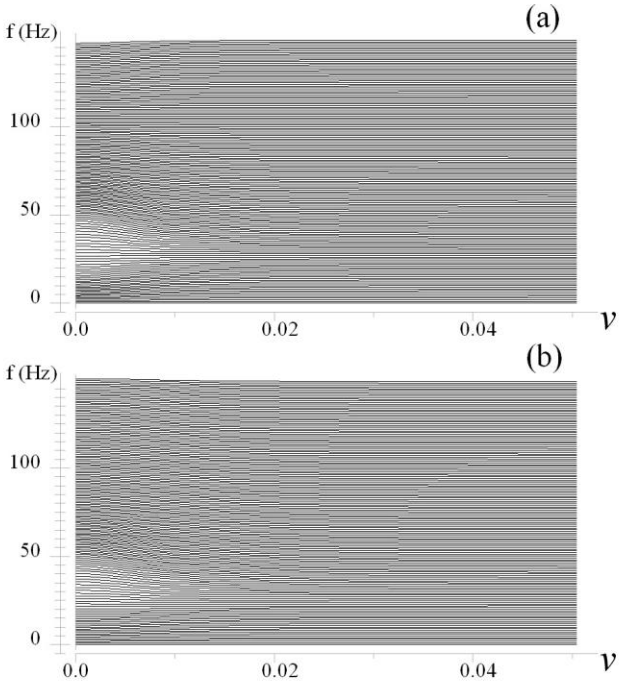

Figure 6

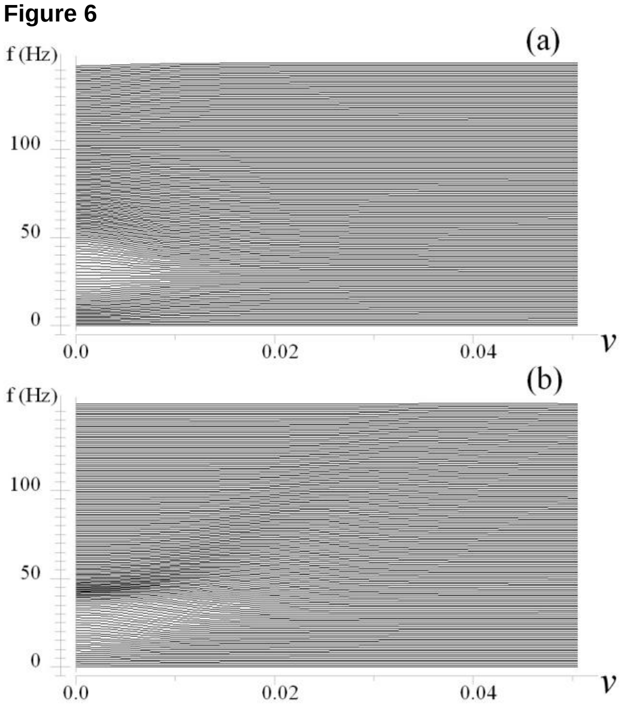

Figure 7

The results presented in Figures 6-8 can illustrate this fact. Here values of a two-dimensional spectrum which have been calculated on the basis of analytical expressions (parts (a) of the Figures) and with use of procedures of the discrete Laplace and Fourier-Bessel transforms are shown, see parts (b) of the Figures. It is visible that for very small $t_1 < 0,00004\sec$ both kinds of spectra practically coincide, see Figure 6. However, even for small changes (an order $0,0001$) a significant distinction starts to appear between these spectra, see Figure 7. Thus distinction in structure of two types of spectra becomes very significant for $t_1 \geq 0,004\sec$, see Figure 8.

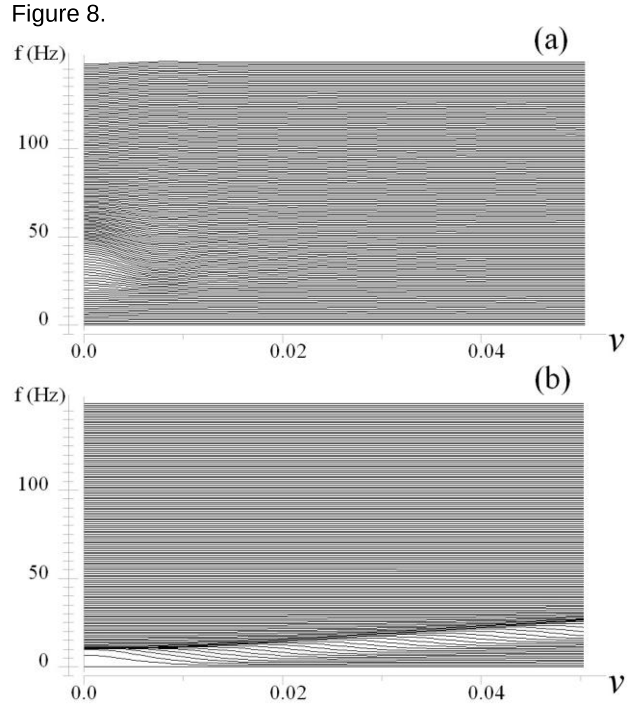

Figure 8

It is necessary to notice that changes of analytically giving spectra in these cases are observed, but are of insignificant character.

An analysis of the influence of α on the structure of the obtained spectra is important for the correct solution of the inverse problem. On use of values α as some regularization parameter, allowing to approximate the exact solution obtained in spectral area with calculated two-dimensional spectra, it was indicated in Mitrofanov, Priimenko and Missagia (2007 (b)). Therefore we have used a series of experiments with the test function for the study of the specified influence.

Figure 9 shows, how the results of the Laplace transform change with variation of α. These results are connected with the second type of function with $\alpha_0 = 10$ and $t_1 = 0$. Thus, the values of α are changed from $0,1$ to 100. It is visible that with increasing of α the structure of $\tilde{U}^*(x, f)$ is changed from the second type of function to the first type with $\alpha_0 = 100$ and $t_1 = 0$, compare corresponding Figures 2(a) and 3(a). This result is fairly obvious if to consider using these parameters in (4). According to these expressions increasing of α will be equivalent to increasing of α_0 in the signal representation.

The experiments with complete two-dimensional spectra have shown that in this case using α also gives the effect of change of the type of the signal noted above. Such change is sufficiently proved by the structure of analytical expressions (6) and (7). But at the same time, the influence of α on the structure of the complete spectrum will be more difficult, taking into account that this parameter enters in $a_1 = \alpha_1 + \alpha\, t_1$, where α interacts with parameters α_1 and t_1. Thus, α can regularize the structure of the two-dimensional spectrum. But this change influences the structure of the impulse and does not lead to full combination of the spectra, constructed using analytical expressions, and expressions, calculated on the basis of discrete analogues of the giving transforms.

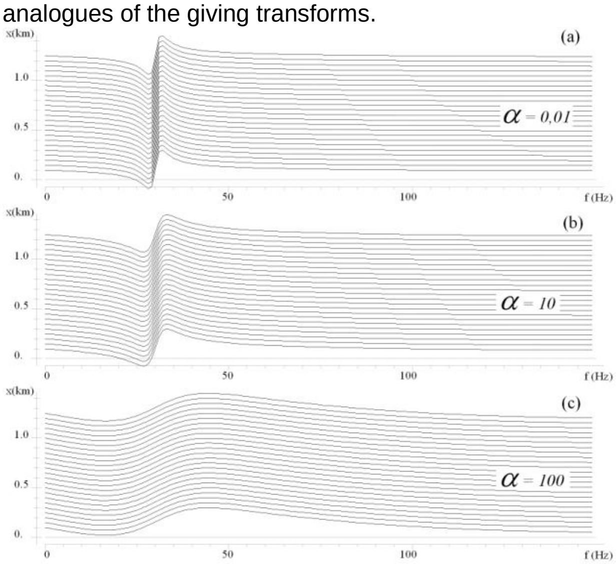

Figure 9

Conclusions

Constructed with use of theoretical function obvious expressions, both for the Laplace transform and for the Laplace and Fourier-Bessel transforms, have enabled the carrying out of a series experiments regarding possible influence of the given transforms on the obtained solution. Also they have given the opportunity to study some properties of discrete analogues of the considered transforms.

Acknowledgments

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