Geometric Theory of Seismic Imaging

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Abstract

In this paper I provide an overview of main concepts and results by S.V.Goldin in the field of geometric theory of seismic imaging. Then I present some recent results on velocity continuation of seismic images developing his ideas.

Introduction

Main areas of seismology that S.V. Goldin have contributed to include: statistic methods of signal detection in seismic traces; inverse kinematic problem for layered media; geometric theory to seismic imaging, physics of the earthquake source.

In this paper we will discuss geometric theory of seismic imaging following Goldin (1998,2003). Theory of contact mappings in seismic imaging was first discussed in details in (Goldin, 1994). Similar ideas were developed in (Hubral et al., 1996; Tygel et al., 1996). Concept of velocity continuation was first introduced by (Fomel, 1994a).

Operators. Let us consider an operator F transforming some input function to another one:

$$F:\ u(\mathbf{x}) \to w(\mathbf{y}), \tag{1}$$

and its adjoint F^*. Here we will consider the case when $\mathbf{x}$ and $\mathbf{y}$ are of the same dimension and F is invertible. We usually assume that $u(\mathbf{x})$ and $w(\mathbf{y})$ contain singularities supported on a piece-wise smooth surfaces Φ and Ψ correspondingly. Popular examples are reflectors present in an image and traveltime surfaces in data. Then we use terms 'migration' for F^* and 'demigration' for F in general sense of displacing singularities in $u(\mathbf{x})$ into those in $w(\mathbf{y})$. Note that most of seismic processing procedures fall into category of operators (1): modeling, migration, offset data transformation, remigration etc.

There are few ways to implement these operators:

1. Boundary-value problems for hyperbolic partial-differential equations (PDEs).
2. Generalized Radon Transform (GRT) integral operators.
3. Fourier Integral Operators (FIOs).

GRT or Kirchhoff type integral operators can be defined as follows:

$$w(\mathbf{y}) = \int a(\mathbf{y},\mathbf{x})\, \delta\big(\phi(\mathbf{y},\mathbf{x})\big)\, u(\mathbf{x})\, d\mathbf{x}\,, \tag{2}$$

where $\phi(\mathbf{y},\mathbf{x}) = 0$ defines summation hypersurfaces.

Inverse scattering theory developed in the framework of FIOs is described in (De Hoop, 2003). FIOs represent a class of operators even more general than GRT:

$$w(\mathbf{y}) = \iint a(\mathbf{y},\mathbf{x},\boldsymbol{\theta})\, e^{i\varphi(\mathbf{y},\mathbf{x},\boldsymbol{\theta})}\, u(\mathbf{x})\, d\mathbf{x}\, d\boldsymbol{\theta}\,, \tag{3}$$

with some properties on $a(\mathbf{y},\mathbf{x},\boldsymbol{\theta})$ and $\varphi(\mathbf{y},\mathbf{x},\boldsymbol{\theta})$ - amplitude and phase function correspondingly.

Below we will consider a relation between these two types of operators.

Contact mappings

Geometric properties of F, i.e. transformation of singularities by this operator, are described by a notion of contact mapping:

$$K:\ (\mathbf{x},\mathbf{n}) \to (\mathbf{y},\mathbf{m}), \tag{4}$$

where contact element is a pair $(\mathbf{x},\mathbf{n})$ or $(\mathbf{y},\mathbf{m})$ defining position and orientation (as shown schematically in Fig. 1); $\mathbf{n}$ are $\mathbf{m}$ are orientation vectors. Note that any point $\mathbf{x}$ (or $\mathbf{y}$) is considered as a family of contact elements with all possible orientations (see Fig. 2,left).

K is a one-to-one, invertible mapping of contact elements. In addition it should satisfy a special "contact" property as illustrated in Fig. 2: point $\mathbf{x}$ can not be mapped to a point $\mathbf{y}$ but to a piece-wise smooth surface $\psi_{\mathbf{x}}$ (called special surface). This property is closely related to a concept of propagation. That is, point source can not move while remaining a point but should emanate energy smearing it along the Huygens surface. Relation of special (Huygens) surfaces to diffraction traveltime curves and isochrones is described in details in (Hubral et al., 1996). Then K can be defined as a mapping of all point sources to corresponding special surfaces:

$$K:\ \mathbf{x} \to \psi_{\mathbf{x}}(\mathbf{y}). \tag{5}$$

A full set of special surfaces uniquely defines a contact mapping K (Goldin, 1998).

The third way to define K is to represent it as a surface-to-surface contact mapping:

$$K:\ \Phi \to \Psi. \tag{6}$$

Contact mapping definition in the form (5) is very natural for developing Kirchhoff type migration-demigration operators based on Born single scattering theory. Form (6) is natural for understanding migration-demigration as a mapping of a traveltime surface to a reflector and vice versa. However definition of K as a contact mapping (4) is the most fundamental one. First, in this form it is identical to the so-called canonical relation that describes propagation of singularities by an FIO (De Hoop, 2003). Second, it corresponds to a concept of the map migration. Third, as illustrated in Fig. 2 and 3 point-to-surface (5) and surface-to-surface (6) mappings are easily derived from (4) (opposite is not true).

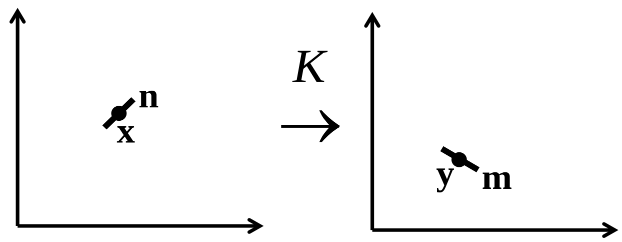


Figure 1: Contact mapping of contact elements.

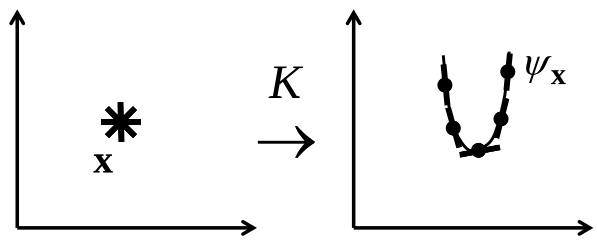


Figure 2: Contact mapping (point-to-surface).

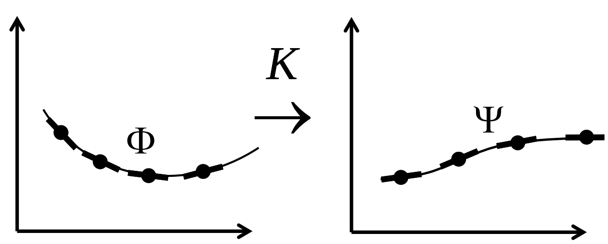


Figure 3: Contact mapping (surface-to-surface).

Contact continuation. Contact continuation is a smooth family of contact mappings K_α parameterized by a scalar α.

Relations between different types of operators

There were a few successful attempts to derive new PDEs implementing contact continuation appearing in seismic imaging: velocity continuation, DMO, remigration (Fomel, 1994; Hubral et al, 1996). However this is a rather restrictive class of operators. Only a few types of contact mappings can be implemented this way (mostly for zero-offset and constant background velocity case).

GRT operators (2) appear naturally in the Born scattering and inverse scattering theory. This is a very natural way for constructing migration-demigration operators. Goldin (1994) showed that every contact mapping K can be implemented as a GRT operator.

What is a relation between an FIOs and a GRT operator? Conceptually FIO is similar to constructing a wavefront by solving eikonal equation while GRT would correspond to constructing it using the Huygens principle.

One can rewrite GRT operator (2) in the following form:

$$w(\mathbf{y}) = \frac{1}{2\pi}\iint a(\mathbf{y},\mathbf{x})\, e^{i\theta\phi(\mathbf{y},\mathbf{x})} u(\mathbf{x})\, d\mathbf{x} d\theta . \qquad (7)$$

One can see that GRT operator (2) appears to be a particular case of an FIO (3). Then it follows that every K can be implemented as an FIO. Note that K describes a canonical relation (propagation of singularities) by an FIO.

The following theorem is true:

Theorem. Every contact continuation K_α can be implemented as a solution to a Cauchy problem for a hyperbolic (pseudodifferential) evolution equation:

$$\left[\partial_\alpha - P(\alpha,\mathbf{x},D_\mathbf{x})\right] u(\mathbf{x},\alpha) = 0 , \qquad (7)$$

$D_\mathbf{x} = \partial/\partial\mathbf{x}$, P - pseudodifferential operator (PsDO):

$$Pu(\mathbf{x},\alpha) = \int p(\alpha,\mathbf{x},\mathbf{k_x}) e^{i\langle \mathbf{k_x},\mathbf{x}\rangle} \hat{u}(\mathbf{k_x},\alpha) d\mathbf{k_x} , \qquad (8)$$

where $\hat{u}(\mathbf{k_x},\alpha)$ is a Fourier transform of $u(\mathbf{x},\alpha)$ in $\mathbf{x}$, $p(\alpha,\mathbf{x},\mathbf{k_x})$ - homogeneous in $\mathbf{k_x}$; $\mathbf{k_x}$ - wave numbers (Fourier duals of $\mathbf{x}$).

To prove this statement we just note that there is an invertible FIO $F(\alpha)$ corresponding to every contact mapping K_α. From (Duchkov et al., 2008) it follows that a smooth family of FIOs $F(\alpha)$ satisfies equation (7).

Propagation of contact elements (singularities) is described by the Hamiltonian:

$$H(\alpha,\mathbf{x},k_\alpha,\mathbf{k_x}) = k_\alpha - p_1(\alpha,\mathbf{x},\mathbf{k_x}) , \qquad (9)$$

where p_1 is a principal symbol of the operator P .

Pseudodifferential operators (PsDO) versus PDEs. Representation of seismic imaging operators in the form of PDEs is attractive because one can implement them using efficient finite-difference algorithms. Evolution equations (7) involving PsDOs represent a more general class of operators. In fact these operators allow extending notion of continuation (in velocity, offset etc.) to the case of general heterogeneous medium. Well-known double-square root operator is an example of such an evolution equation. Unfortunately PsDOs are much more expensive from a computational point of view. An efficient way evaluate them is to use generalized screen methods.

Continuation by evolution equation

Deriving continuation operators (Fomel, 1994; Tygel et al., 1996) it is possible to start by composing demigration (F) and migration (F^*) operators:

$$C_\alpha = F^*(\alpha) F(\alpha_0) , \qquad (10)$$

where α can stand for a velocity model perturbation, offset or something else.

Following (Duchkov et al., 2008) we can find operator P in the evolution equation (7) as follows:

$$P(\alpha) = \partial C_\alpha = \partial_\alpha F^*(\alpha) F(\alpha_0) \quad (11)$$

Right hand side in (11) is a manifestation of the 'remigration' strategy. In this case continuation is realized by a composition of two operators. In order to get a 'true' continuation one needs to transform this composition into a single operator P.

Derivations of this type for GRT operators can be found in (Hubral et al., 1996). Derivation of evolution equtions in a framework of FIOs can be found in (Duchkov et al., 2008). Below we present Hamiltonians describing continuation in the context of common-offset (CO) migration.

Velocity continuation of CO image. Hamiltonian for common-offset remigration operator (for the constant velocity case):

$$H(v,x,z,k_v,k_x,k_z) = k_v + \frac{k_x^2 + k_z^2}{2vk_x^2 k_z}\left[z(k_x^2 - k_z^2) + \sqrt{(2hk_x k_z)^2 + z^2(k_x^2 + k_z^2)^2} \right] \quad (12)$$

h - offset that is a constant parameter here; v - constant background velocity used as an evolution parameter for the continuation.

Note that due to equation (9) from the Hamiltonian we automatically get a principal symbol of the evolution equation (7). However, the Hamiltonian itself is all what we need here as we consider only the geometry of propagation of singularities here. Also note that a Hamiltonian corresponding to a PDE should be a polynomial in k_v, k_x and k_z. Thus we see that the Hamiltonian (12) corresponds to a pseudodifferential equation of type (7).

In Fig. 4 we illustrate both – notion of contact elements (short bold plates) and velocity rays corresponding to (12) (thin lines). Envelope of contact elements forms the reflector image. Initial parabolic reflector corresponds to $v = 1$ km/s. According to the velocity continuation concept, reflector images propagate as pseudo-fronts with changing velocity v. We see that the Hamiltonian (12) describes anisotropic propagation of the image: velocity rays are not normal to initial image (initial pseudo-front). One can see that the velocity rays form a caustic. For $v = 1.06$ km/s we see two cusps in the final image that appeared due to caustics formed by the velocity rays.

Isochron propagation in CO migration. Isochrons are usually considered as impulse response of a migration operator to one non-zero value in data. Thus they are naturally associated with 'fronts' (Huygens surfaces) corresponding to a 'point source' in data. This interpretation rises a reasonable question: is there an evolution equation describing propagation of singularities?

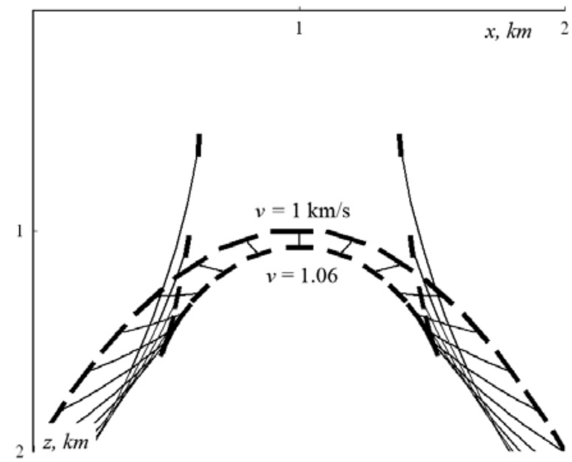


Figure 4: Velocity continuation for parabolic reflector. Contact elements (bold) and velocity rays (thin lines).

Below we provide a Hamiltonian describing propagation of isochrones in case of constant velocity v (two-way time t is used as an evolution variable):

$$H(x,z,\omega,k_x,k_z) = \omega - \frac{v}{zk_x\sqrt{2}}\left[\frac{\sqrt{Q_- Q_+}}{\sqrt{Q_-} + \sqrt{Q_+}}\right], \quad (13)$$

$$Q_\pm = z^2\left[k_x^2 + k_z^2\right]^2 + \left[2hk_x k_z \pm z(k_x^2 - k_z^2)\right] q_\pm,$$

$$q_\pm = 2hk_x k_z \pm \sqrt{4h^2 k_x^2 k_z^2 + z^2(k_x^2 + k_z^2)^2},$$

h - offset that is a constant parameter here; ω - frequency (Fourier dual to time t).

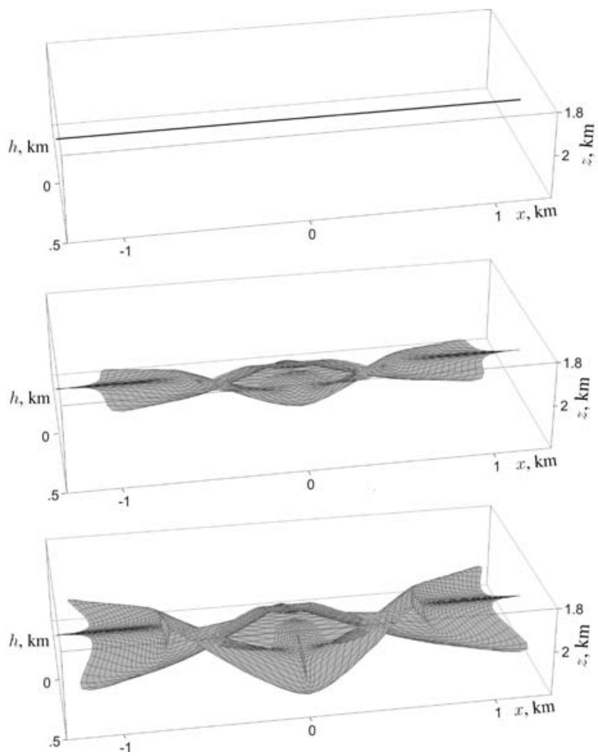
Extended imaging operators.

The whole theory described in this paper was based on assumption that $\mathbf{x}$ and $\mathbf{y}$ have equal dimension. This property is also essential for proving invertibility of a migration operator F^* and existence of a corresponding contact mapping K. For this reason results in (Fomel, 1994; Goldin, 1994; Hubral et al., 1996, Duchkov et al., 2008) are typically restricted to zero-offset or common-offset case and thus assume absence of caustics.

Recently it was shown that the same theory of constructing evolution equations for data or image continuation can be applied to pre-stack imaging. In this case one should extend imaging operators (De Hoop, 2003) so that data $u(\mathbf{x}) = u(s,r,t)$ (s - source coordinate, r - receiver coordinate, t - time) are mapped to extended image $w(\mathbf{y}) = w(x,z,h)$ (x - horizontal coordinate, z - depth, h - subsurface offset). In this case dimensions of $\mathbf{x}$ and $\mathbf{y}$ match again and one can prove invertibility of migration-demigration operators under conditions allowing caustics.

Considering heterogeneous media we can choose a line in a space of possible background velocities $v(\alpha; x,z)$ parameterized by a scalar parameter α. Then one can consider the velocity continuation of an extended image

$w(\alpha, x, z, h)$ in α. In this case it is still true that there exists an evolution equation (7) for the velocity continuation (in α) (Duchkov et al., 2009). It is impossible to write it out corresponding Hamiltonian explicitly but it is possible to evaluate it. In Fig. 6 one can see a velocity model with a smooth low velocity lens and a horizontal reflector at the depth $z = 2$ km. Extended image for a 'correct' velocity is shown at the top of Fig.5. Then we start removing the lens from the migration velocity. The extended image starts evolving like a pseudo-front forming typical cusp caustics.



Fugure 5: Evolution of the extended image of a horizontal reflector (from top to bottom) with background velocity perturbation.

Conclusions

In this paper I have presented main concepts developed by S.V. Goldin in order to study operators of seismic imaging. These include notion of contact mapping and contact continuation. It is a very explicit way to describe the geometry of imaging operators which map traveltimes and local slopes into reflector depth and local dip.

Another important question considered was a way of implementing imaging operators given the geometry, e.g. full description of a contact mapping. Every contact mapping can be implemented by a GRT operator. Every contact continuation can be implemented as a solution to a hyperbolic pseudodifferential evolution equation.

I have provided a few examples of Hamiltonians describing propagation of singularities during a velocity continuation of seismic (extended) images.

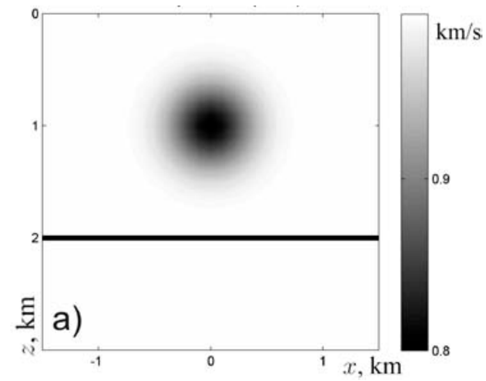


Figure 6: Smooth velocity model with low velocity lens and horizontal reflector.

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