Numerical solution of the acoustic wave equation by the rapid expansion method (REM) - A one step time evolution algorithm

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This paper was prepared for presentation at the 11th International Congress of The Brazilian Geophysical Society held in Salvador, Brazil, August 24-28, 2009.

SUMMARY

We revisit a numerical solution for the time evolution of the acoustic wave equation where the wave field can be advanced in time in one large step, the rapid expansion method, or REM. This method has the advantage of being numerically stable even for large time steps and when combined with a pseudo spectral Fourier method for the spatial derivatives results in a highly accurate solution for the propagating wave fields. We show that the number of terms in the integration that produces the time snap shots can be significantly reduced making the method computationally attractive for high resolution simulations.

INTRODUCTION

The acoustic wave equation is usually solved using finite differences, finite elements or Fourier based methods. These direct solution methods typically feature discrete temporal and spatial differentiation approximations. Spatial approximations are accomplished with different degrees of accuracy. Temporal derivatives are typically approximated by some form of differencing based on Taylor series expansions. The most common practice has been to use second-order differencing in time. In particular for higher spatial approximations, very small time steps are required to obtain satisfactory results and even then some degree of numerical dispersion is always present (Kosloff and Baysal (1982)).

In the work presented by Tal-Ezer et al. (1987) a new time integration technique for solving the acoustic wave equation was introduced based on a Chebyshev expansion of the formal solution of the wave equation. This technique is very accurate and presents an attractive alternative to second-order temporal differencing when spatially accurate methods like high-order finite differences or the pseudo spectral Fourier method are employed for the spatial derivatives. Originally, the method was derived for a coupled first-order system in time, instead of the second-order wave equation. Consequently, this method was not significantly faster than methods based on temporal differencing, and depending on the implementation required additional storage (Tal-Ezer et al. (1987)).

Two years later, Kosloff et al. (1989) introduced the rapid expansion method (REM), based on the solution of the second-order wave equation and used concepts similar to Tal-Ezer's original work. Their approach improved both the speed and the storage requirements: it is twice as fast as at the original Tal-Ezer method; and, it requires only half as much storage (Kosloff et al. (1989)).

REM can be used to propagate the wave field forward in time in one large step, since it is numerically stable for any time step size selected. Further, when combined with a pseudo spectral Fourier based spatial operator no numerical dispersion is present and all temporal and spatial frequencies are modeled correctly.

THEORY AND METHOD

In an acoustic medium in which the density is constant but the velocity can vary both vertically and laterally, the wave propagation in a 2-D acoustic medium is governed by the following acoustic wave equation

$$\frac{\partial^2 p}{\partial t^2} = -L^2 p + S \tag{1}$$

where $p(x,z,t)$ denotes the pressure field, t denotes time and x and z denote the Cartesian coordinates and $S = S(x,z,t)$ represents a source term. The operator $-L^2$ is given by:

$$-L^2 = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \tag{2}$$

where $c = c(x,z)$ is the acoustic velocity. The basis for the REM method is the formal solution of equation (1), without the source term, and with the initial conditions

$$\frac{\partial p}{\partial t}(t=0) = \dot{p_0} \qquad and \qquad p(t=0) = p_0 \tag{3}$$

and is given by the following expression:

$$p_t = \cos(Lt)\, p_0 + \frac{sin(Lt)}{L}\, \dot{p_0} \tag{4}$$

Adding to (4) the solution for the term p_{-t} then we get

$$p_t = -p_{-t} + 2\cos(Lt)\, p_0 \tag{5}$$

However, the cosine function in the equation (5) can be also written as:

$$\cos(Lt) = \frac{1}{2}\left(e^{-iLt} + e^{iLt}\right). \tag{6}$$

The REM is obtained with the following expansion of the cosine function (Tal-Ezer (1986)):

$$\cos(L\Delta t) = \sum_{k=0;\,(k\,even)}^{M} C_k J_k(\Delta t\, R)\, Q_k\left(\frac{iL}{R}\right) \tag{7}$$

where $C_0 = 1$ and $C_k = 2$ for $k \geq 1$. J_k represents the Bessel function of order k and $Q_k(w)$ are modified Chebyshev polynomials and R is a scalar larger than the range of eigenvalues of L.

This sum (7) contains only even polynomials, in contrast to the original Tal-Ezer method. Moreover, $Q_k(w)$ for even k contains only powers of $-L^2$, which is the basic operator $c(x,z)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)$. Consequently, we need a recursion for $Q_k(w)$ with only even terms. In this case, it is more convenient to use the alternative relation

$$Q_{k+2}(w) = (4w^2+2)\,Q_k(w) - Q_{k-2}(w). \qquad (8)$$

The recursion is initiated by:

$$Q_0(w) = 1 \qquad and \qquad Q_2(w) = 1+2w^2.$$

Source term

Assuming that the source is separable according to $S(x,z,t) = G(x,z)H(t)$,where all time dependence is contained in $H(t)$, the formal solution to (1) (Kosloff et al. (1989)) is

$$p(x,z,t) = \left[\int_0^t \frac{\sin(L\tau)}{L}\,H(t-\tau)\,d\tau\right] G(x,z) \qquad (9)$$

Following the work of Tal-Ezer (1986) the function $\frac{\sin(L\tau)}{L}$ is represented by the following Chebyshev expansion:

$$\frac{\sin(L\tau)}{L} = \sum_{k=1;\,(kold)}^{M} C_k\,\frac{J_k(tR)}{R}\left(\frac{R}{iL}\right) Q_k\left(\frac{iL}{R}\right) G(x,z) \quad (10)$$

or,

$$p(x,z,t) = \sum_{k=1;\,(kold)}^{M} C_k\,b_k\left(\frac{R}{iL}\right) Q_k\left(\frac{iL}{R}\right) G(x,z) \qquad (11)$$

where,

$$b_k = \int_0^t \frac{J_k(\tau R)\,H(t-\tau)}{R}\,d\tau$$

For 2D wave propagation R is approximated given by $\pi c_{max}[\frac{1}{dx^2}+\frac{1}{dz^2}]^{1/2}$ with c_{max} the highest velocity in the grid and dx and dz are the grid spacing (Kosloff et al. (1989)). The sum in (7) is known to converge exponentially for $k > tR$ and, therefore, the summation can be safely truncated with k value slightly greater then tR (Tal-Ezer (1986)).

Equation (8) with w replaced by $\frac{iL}{R}$ form the basis of the REM approach. It is important to note that only even powers of the operator L are required. This is important because only these terms are directly defined.

In the REM approach, unlike the original Tal-Ezer method, the sum in (7) is only over even terms and, therefore, the number of required functional evaluations is half the number of evaluations required in the original Tal-Ezer approach. We can therefore conclude that the REM should be twice as fast as the original Tal-Ezer method and methods based on second-order temporal differencing.

We noted that the summation can be truncated for $k > tR$ but we also note that for larger times and for small k, the contributions to the integral are small and insignificant. Figure 1 shows the values of $J_k(tR)$ as a function of k and t. The threshold where the integration weights are insignificant can be described by a line of constant slope and terms less than this line can be eliminated without degrading the results.

The basic storage requirements of the REM are about the same as the requirements of temporal differencing. However, with REM depending on the implementation, all intermediate results might be stored until the end of the computations or a rolling buffer of terms can be employed. Therefore, when many snapshots or time sections are desired, the storage requirements can become large and the method of implementation is critical to make the method attractive. But, for forward modeling of known receiver configurations, only the terms at the spatial positions of the receivers need be saved greatly reducing the storage requirements.

EXAMPLES

We illustrate the REM with a 2D example for the EAEG salt model. The grid spacing in x and z was .02 km. The source was a delta function in time. Figure 2 shows Chebyshev polynomial term 850 where the source was located at 4.0 km and just below the free surface. A pseudo spectral method was used to calculate the spatial derivatives. We note that even though the response of this term appears like a propagating wave, time is not known at this point in the calculation. The wave like character shown results from the repeated recursive application of the Laplacian used to generate the Chebyshev expansion terms. In this example a free surface was used. We see the surface multiple, the waves entering into the salt body and reflection from the top of the salt in addition to reflections from layers above the salt body.

Figure 3 shows a time snap shot at 1.5 s from a similar modeling experiment but now the source, which was again a delta function in time was located at 5.0 km. Free surface multiples were suppressed by using an absorbing boundary just above the source. Here we note that the wave has passed through the salt body and reflections from the top and bottom of the salt are present. Only terms 561 to 785 (a total of 225 terms) were used to generate this snap shot. The total CPU time on a windows PC was 26 s and we note that additional snap shots could have been generated with just the time required for the integration since all the Chebyshev polynomials were generated in advance.

Figure 4 shows the seismic data for a delta function source in time located at 7.0 km. Data were recorded at the surface. For display purposes AGC was applied and a high pass filter of 8 to 125 hz. The data were calculated at a .004s interval. Figure 5 shows the same data in detail for the wide angle reflections and refractions. These low phase velocity events show no grid dispersion. In this display no AGC or band pass filtering were applied.

Using a pseudo-spectral method for the spatial operator and a 2nd order finite difference scheme in time, for the 2D case and considering $dx = dz$, the stability limit is given by $\alpha = c_{max} dt/dx \leq 2/\pi$. For the salt dataset used here, with $c_{max} = 4.5 km/s$ and $dx = dz = .02km$, for $\alpha < 0.2$, we need to have $dt \leq 8.889x10^{-4}s$. The maximum frequency for modeling without dispersion will be $f_{max} \leq \frac{v_{min}}{2dx}$. If we take $v_{min} = 1.5km/s$, we have $f_{max} \leq 37.5Hz$. Considering the approach presented here by the REM, with $f_{max} = 125Hz$ we need to have $dx = dz = .006km$, consequently $dt \leq 2.667x10^{-4}s$. To run this modeling problem for a 4s maximum time computation the required number of time steps (t_{max}/dt) is 14,998 times steps. However, the number of laplacian calculations needed by the REM approach is $M > t_{max}R/2 \approx 2000$, which need to be evaluated to compute the whole seismogram. Thus for high fidelity results the computation cost for REM is much less compared to a 2nd order finite difference time operator.

CONCLUSIONS

We have investigated the use of the REM for seismic modeling and shown that the results produced are of high quality for a complex salt model. We have shown that the number of terms used in the integration to produce the time response can be limited by specifying the first and last term to be used based on the magnitude of the Bessel functions for the time of interest. We also note that storage can be significantly reduced by saving the required Chebyshev polynomials only for the spatial positions which are of interest, e.g. at the receiver positions. This greatly reduces the storage requirement, especially for 3D implementations and further reduces the computation time. We suggest that this method (which can be easily extended to elastic and anisotropic propagation) is of particular interest when coupled with a pseudo spectral Fourier method for the spatial derivatives since high numerical accuracy can be achieved with little or no numerical problems, such as grid dispersion.

ACKNOWLEDGMENTS

This work was made possible with funding from the King Abdullah University of Science and Technology (KAUST). We are grateful for this financial support.

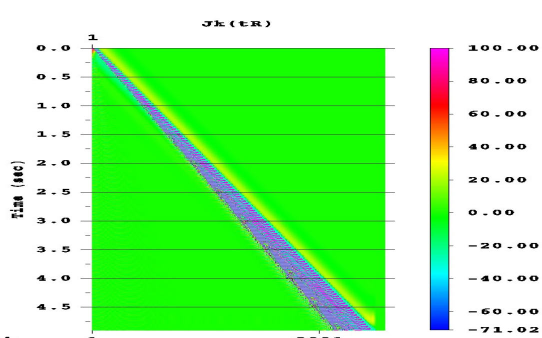

Figure 1: Plot of $J_k(tR)$ for a maximum velocity of 4.5 km/sec and a grid spacing of .02 km in x and z. A gain of 100 was applied for display purposes. Most of the values except for a narrow cone are very close to zero and do not contribute in any significant way to the integral.

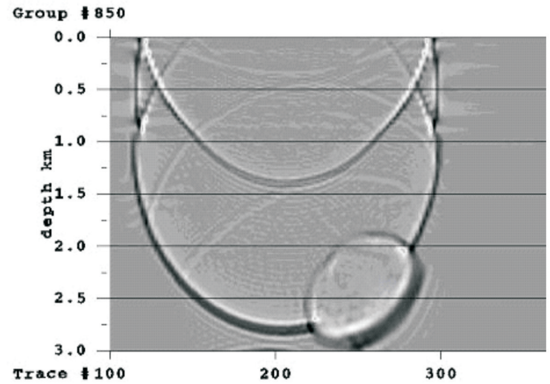

Figure 2: Plot of term 850 of the Chebyshev expansion for the EAEG salt model. For this term we see the wave front reflecting from and entering into the salt body (lower right). In addition we see reflections from above the salt returning toward the free surface and the free surface multiple.

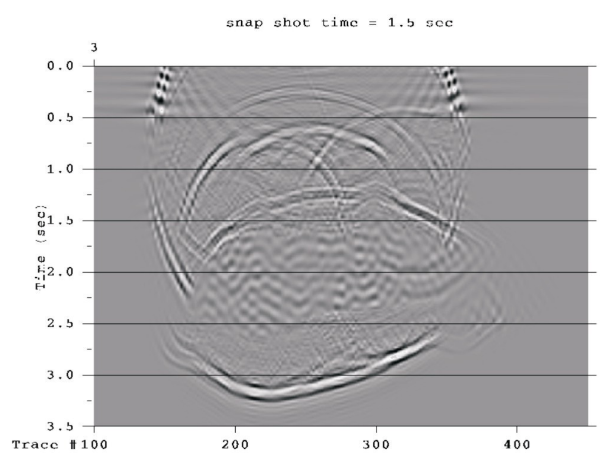

Figure 3: Snap shot of the wavefield at T=1.5s calculated by integrating 225 Chebyshev polynomial terms. Note that in this example the wave field has left the bottom of the salt and the presence of reverberations within the salt body.

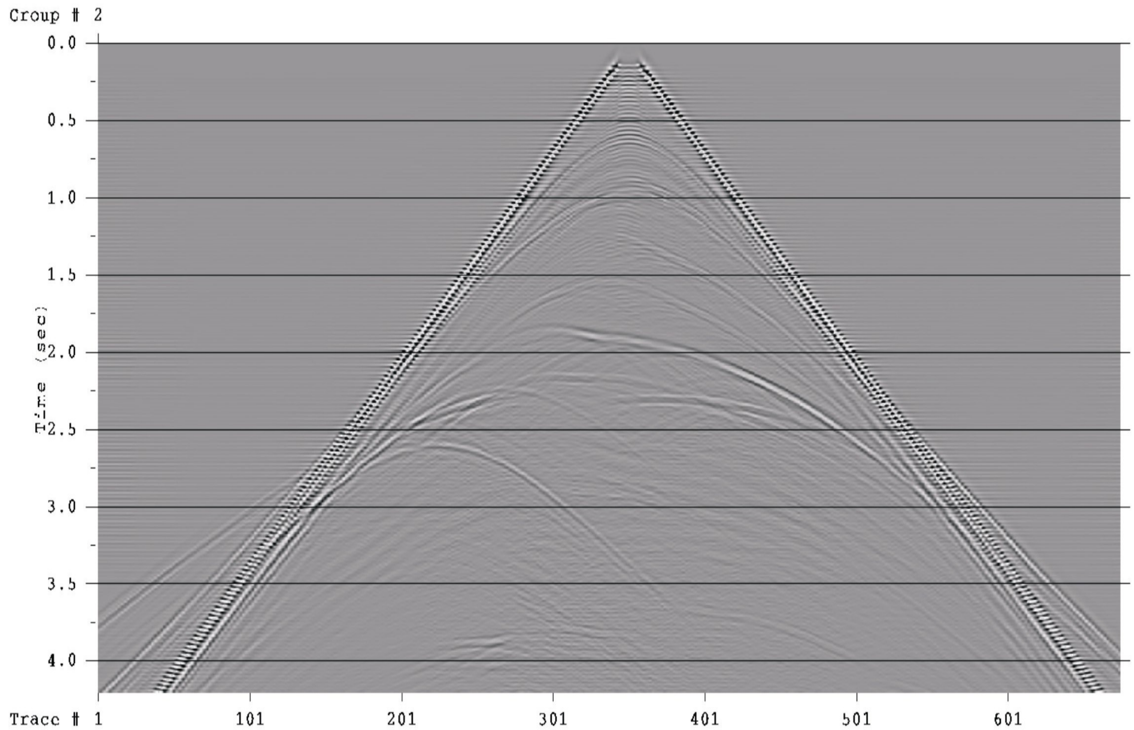

Figure 4: Seismic data recorded just below the surface for a delta function source in time at 7.0 km. For display purposes AGC was applied and the data were high pass filtered from 8 to 125 hz. No grid dispersion is apparent. Note only every other trace is displayed here so that the trace spacing is .04 km.

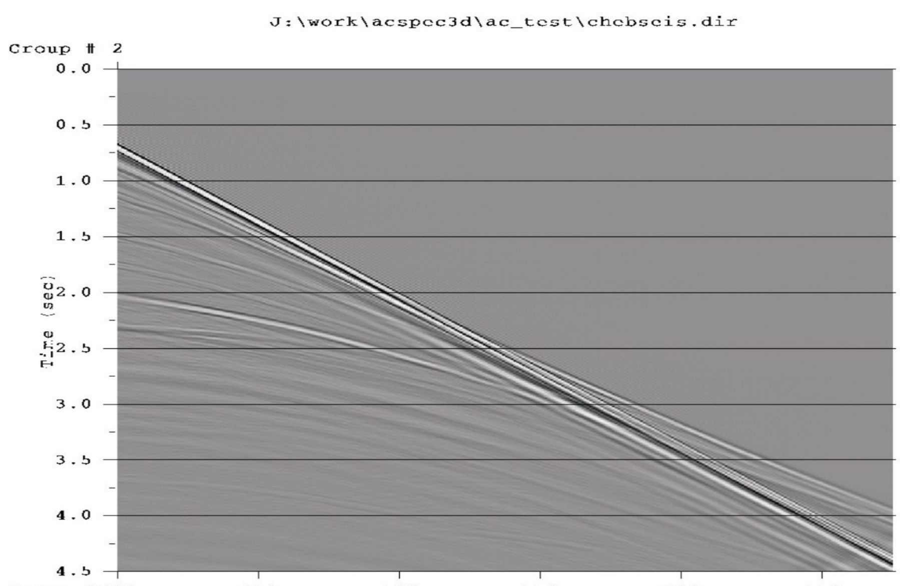

Figure 5: Detail display for the same data as Figure 4 showing the wide angle reflections and refractions. The low phase velocity arrivals show no dispersion. In this display no AGC or band pas filtering were applied.

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