Rapid expansion method (REM) for time-stepping in reverse time migration (RTM)

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SUMMARY

We show that the wave equation solution using a conventional finite-difference scheme, derived commonly by the Taylor series approach, can be derived directly from the rapid expansion method (REM). After some mathematical manipulation we consider an analytical approximation for the Bessel function where we assume that the time step is sufficiently small. From this derivation we find that if we consider only the first two Chebyshev polynomials terms in the rapid expansion method we can obtain the second order time finite-difference scheme that is frequently used in more conventional finite-difference implementations. We then show that if we use more terms from the REM we can obtain a more accurate time integration of the wave field. Consequently, we have demonstrated that the REM is more accurate than the usual finite-difference schemes and it provides a wave equation solution which allows us to march in large time steps without numerical dispersion and is numerically stable. We illustrate the method with post and pre stack migration results.

INTRODUCTION

The acoustic wave equation has been widely used for modeling and reverse time migration of seismic data. In pre-stack reverse time migration the source wave field is forward extrapolated and the recorded data are backward extrapolated in time. The image is constructed by cross correlating the extrapolated source and receiver wave fields at each time step. The finite-difference method has long been the common approach used to extrapolate the wave fields forward and backward in time. But to ensure high quality results, accurate approximations are required for both the spatial and time derivatives (Dablain (1985)). This is usually achieved numerically by using either a very fine computation grid or very long finite-difference operators. Otherwise, numerical errors, such as grid dispersion, will be present in the data and will contaminate the signals of interest (Liu et al. (2008)).

For increased spatial accuracy, the Fourier pseudo-spectral method can be used to compute the spatial derivatives but in most implementations the time derivative remains as a finite difference operator. The Rapid Expansion Method (REM) proposed by Kosloff et al. (1989) can instead be used to obtain a more accurate time integration of the wave equation. In the REM Chebyshev polynomials are used to expand the cosine operator which appears in the exact solution of the wave equation. This solution method using different initial conditions can be used to extrapolate wave fields forward or backward in time from any time step.

Here we expand the cosine operator from the exact solution of the acoustic wave equation in the same way as that used by the REM approach. After that, we note that the result of the expansion has the same form as the Taylor series expansion when we use a specific analytical expression for the Bessel function. Then we verify that if we consider only two terms in the REM it reduces to the same equations used for the second order finite difference time approximation. The use of more terms in the REM results in a procedure that is numerically stable even for large time steps. When REM is combined with a pseudospectral method for the spatial derivatives, a highly accurate, numerically stable result can be obtained with less computation than a conventional finite difference approach would require to achieve the same level of accuracy.

THEORY

Acoustic wave equation - An exact solution

For acoustic wave propagation, the governing equation has the form

$$\frac{\partial^2 p}{\partial t^2} = -L^2 p; \quad \text{with} \quad -L^2 = c^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) \tag{1}$$

The formal solution of equation (1), with the initial conditions $\frac{\partial p}{\partial t}(t=0) = \dot{p_0}$ *and* $p(t=0) = p_0$ is given by the following expression:

$$p(t) = \cos(Lt)\, p_0 + \frac{sin(Lt)}{L}\, \dot{p_0} \tag{2}$$

The wavefields $p(t+\Delta t)$ and $p(t-\Delta t)$ can be evaluated by equation (2). Adding these two wavefields results in:

$$p(t+\Delta t) + p(t-\Delta t) = 2\cos(L\Delta t)\, p(t) \tag{3}$$

Now if we take for $\cos(L\Delta t)$ its second $(1 - \frac{(L\Delta t)^2}{2})$ and fourth order $(1 - \frac{(L\Delta t)^2}{2} + \frac{(L\Delta t)^4}{24})$ Taylor series expansion, we obtain

$$p(t+\Delta t) - 2\,p(t) + p(t-\Delta t) = -\Delta t^2 L^2\, p(t) \tag{4}$$

and

$$p(t+\Delta t) - 2\,p(t) + p(t-\Delta t) = -\Delta t^2 L^2\, p(t) + \frac{\Delta t^4}{12} L^4\, p(t) \tag{5}$$

which are the second-order and fourth-order standard finite-difference schemes (Etgen (1986); Soubaras and Zhang (2008)).

The Rapid Expansion Method (REM)

As presented by Kosloff et al. (1989) in the Rapid Expansion Method the cosine function given by equation (3) can be also expressed in the following form:

$$\cos(L\Delta t) = \sum_{k(even)}^{\infty} C_k J_k(\Delta t R) Q_k\left(\frac{iL}{R}\right) \tag{6}$$

where $C_0 = 1$ and $C_k = 2$ for $k \geq 1$. J_k represents the Bessel function of order k and $Q_k(w)$ are modified Chebyshev polynomials. Since (6) contains only even polynomials, it is more convenient to use the alternative relation

$$Q_{k+2}(w) = (4w^2+2)\,Q_k(w) - Q_{k-2}(w) \tag{7}$$

The recursion is initiated by:

$$Q_0(w) = 1 \qquad and \qquad Q_2(w) = 1+2w^2$$

For 2D wave propagation the value of R is approximated given by $R = \pi c_{max}\sqrt{\frac{1}{dx^2}+\frac{1}{dz^2}}$, with c_{max} the highest velocity in the grid and dx and dz are the spatial grid spacing (Tal-Ezer et al. (1897)). The sum in (6) is known to converge exponentially for $k > \Delta t R$ and, therefore, the summation can be safely truncated with a k value slightly greater then $\Delta t\, R$ (Tal-Ezer et al. (1897)).

Finite-Difference solution - Special case of REM

Now we can rewrite equation (3) again, but we replace the cosine function by equation (6), thus we have:

$$p(t+\Delta t) + p(t-\Delta t) = 2\left(\sum_{k(even)}^{\infty} C_k J_k(\Delta t R) Q_k\left(\frac{iL}{R}\right)\right) p(t) \tag{8}$$

The Chebyshev polynomials, present in the equation (8), can also be rewritten in terms of $Q_2(w)$ as

$$\begin{aligned}
Q_0(w) &= 1\\
Q_2(w) &= 1+2w^2\\
Q_4(w) &= 2Q_2^2(w) - Q_0(w) = 1+8w^2+8w^4\\
Q_6(w) &= 4Q_2^3(w) - 3Q_2(w) = 1+18w^2+48w^4+32w^6\\
Q_8(w) &= 8Q_2^4(w) - 8Q_2^2(w) + Q_0(w) =\\
&= 1+32w^2+160w^4+256w^6+128w^8
\end{aligned} \tag{9}$$

The Bessel functions, considering the argument z sufficiently small, can be approximated by the following relation (Abramowitz and Stegun (1965))

$$|J_k(z)| = \frac{|z|^k}{2^k\,k!} \tag{10}$$

Substituting equation (9) and the Bessel functions for its approximate value (eq. 10) into the equation (8) and considering only the contribution of the terms in $(wz)^n$ for $n = 0, 2, 4\cdot$, we obtain:

$$\begin{aligned}
p(t+\Delta t) + p(t-\Delta t) &= 2\left(1+\frac{z^2}{2}w^2+\frac{z^4}{24}w^4+\frac{z^6}{720}w^6\right.\\
&+ \left.\frac{z^8}{40320}w^8+\cdots\right)p(t)
\end{aligned} \tag{11}$$

where $w = \frac{iL}{R}$ and $z = \Delta t\, R$.

Now, consider the first two terms of equation (11), which means we are using only terms up to Δt^2, we then obtain:

$$p(t+\Delta t) + p(t-\Delta t) = 2\left(1+\frac{z^2}{2}w^2\right)p(t) \tag{12}$$

and substituting w by $\frac{iL}{R}$ and $z = \Delta t\, R$, we get:

$$p(t+\Delta t) - 2\,p(t) + p(t-\Delta t) = -\Delta t^2 L^2\, p(t) \tag{13}$$

Then the approximation of the cosine function using the Chebyshev polynomials, results in the second-order finite difference in time scheme as given before by (4).

In the same way, the 4th order approximation is obtained from (11) if we consider terms up to Δt^4, then it is given by:

$$p(t+\Delta t) - 2\,p(t) + p(t-\Delta t) = -\Delta t^2 L^2\, p(t) + \frac{\Delta t^4}{12} L^4\, p(t) \tag{14}$$

We can also rewrite the equation (14) in the following form:

$$\frac{1}{\Delta t^2}\left[p(t+\Delta t) - 2\,p(t) + p(t-\Delta t) - \frac{\Delta t^4}{12}L^4\,p(t)\right] = -L^2\,p(t) \tag{15}$$

If now we compare equation (15) with equation (1), we notice that the term on the left is the fourth order approximation for the second order derivative in time (Dablain (1985))

$$\frac{\partial^2 p}{\partial t^2} = \frac{1}{\Delta t^2}\left[p(t+\Delta t) - 2\,p(t) + p(t-\Delta t) - \frac{\Delta t^4}{12}\frac{\partial^4 p}{\partial t^4}\right] \tag{16}$$

where

$$-L^4\,p(t) = L^2\frac{\partial^2 p}{\partial t^2} = -\frac{\partial^2}{\partial t^2}(-L^2 p) = -\frac{\partial^4 p}{\partial t^4} \tag{17}$$

So the L^4 operator term has been replaced by $\partial^4/\partial t^4$.

We conclude that if we use the REM, and retain only the first few Chebyshev terms we can obtain an approximate solution for the wave equation that is more accurate than the conventional Taylor series expansion used to solve (1) with the finite-difference scheme given by equations (3) and 4).

EXAMPLE AND DISCUSSION

Laplacian computation and Stability

In the conventional Fourier pseudo spectral scheme using second order finite differences in time the Laplacian operator L^2 is computed only once for each time step. Normally, in the REM approach the Laplacian must be computed severals times for each time step extrapolation. The number of term in the $cos(Ldt)$ expansion is determine by $M > Rdt$ where R depends on the maximum velocity and the spatial grid (dx,dz). In the Fourier method to guarantee stability and accuracy $\alpha = \frac{c_{max}dt}{dx} < 0.2$. As a test example, we consider a salt dataset where $c_{max} = 4.480km/s$ and $dx = dz = .012km$ (Figure 2(a) - velocity model). To migrate this dataset we need to have it sampled in time at .001 s. But using the REM approach for the original salt dataset, which is sampled at .008 s, the REM will require 8 Chebyshev polynomials or 7 Laplacian computations for each time step extrapolation, as indicated by the first curve in Figure 1. If we resample the data to .002 s, we need to calculate 2 Laplacians based on the criteria indicated. But considering the top curve of Figure 1, which is based on the criteria that the magnitude of the Bessel function be less than 0.001, well need to compute 6 Chebyshev terms for each time extrapolation. The migration results for the .002 s resampled data are shown in Figure 2(b) and Figure 2(c), which were obtained with 3 and 6 polynomials terms or 2 and 5 Laplacian computations for each time step extrapolation, respectively. The result shown in Figure 2(b) was obtained with 3 terms (Q_o, Q_2, Q_4), which is a 4th order approximation in time. This result is now reasonable, but the results with 6 terms (Figure 2(c)) produced a better image for the the salt flank. The result show in Figure 2(d), using the original dataset which is sampled at .008 s, was also obtained with 8 terms. This result is almost identical with Figure 2(c), and it has better signal level in the sub-salt part of the image and was obtained with less computation cost compared to the result for the .002 s dataset computed with 6 terms.

Pre-stack migration result

We now illustrate the use of the REM method for the pre stack deth imaging of a 2D line from the EAEG salt model. The forward and backward time extrapolations were done for .008 sec time steps. The spatial sampling intervals were .02 km in x and z. The input data consisted of 35 shots. The first shot was located at 3.0 km and then every .2 km along the line. 675 receivers, every .02 km along the top of the section, for every shot were used in the RTM. Each shot record was individually migrated; common image gathers were formed and then stacked to produce the final image. A pseudo spectral method was used to compute the spatial derivatives. For the source time response a delta function was used. To be conservative, 16 Chebyshev polynomial terms were used for every time step. This was decided based on the magnitude of the Bessel function being greater than .0001. (Less stringent criteria would lead to fewer terms and improved performance.) Figure 3 shows the stacked common image gathers. For display purposes a high pass filter was used. For this problem a 2nd order time finite difference scheme would require that the wave field snap shots be computed every .002s for a stable result.

We note here that for REM RTM little additional storage was required compared to a 2nd order finite difference solution. This is because the Chebyshev polynomials were integrated as they were computed for both the source and receiver extrapolations.

Figure 4 compares two REM RTM results. On the left the image was formed using only 8 Chebyshev polynomials while on the right 16 were used. All other parameters and data were identical to that of Figure 3. Figure 4 (right) is the same data as Figure 3 but displayed in more detail. We can see that above the salt the results are for all practical purposes identical. But beneath the salt the more accurate time calculation Figure 4 (right) results in more detail and better signal to noise ratio for the sub salt events. However, the differences are overall small and in many cases the factor of 2 computational speed up may be more important. For the 8 polynomial example the time per shot was 5.6 minutes while for the 16 polynomial case the time was 11.8 minutes on a Linux PC.

Finally, we note that for the source calculation we could have calculated all the Chebyshev polynomials in advance and then used these to generate the source snap shots for any time needed, whenever we need them for imaging. This may have some advantages but requires the storage of all the Chebyshev polynomials.

CONCLUSIONS

We have successfully applied the rapid expansion method for time stepping to the reverse time migration of two synthetic salt datasets. We used an exact solution of the acoustic wave equation and showed that the rapid expansion method can be used directly and suggested 2 ways to limit the number of terms required in the integration of the Chebyshev polynomials. Then, by using an analytical approximation for the Bessel functions, for sufficiently small time steps, we obtained different approximations for the second order finite difference time derivative. We showed that as we use more terms in the Chebyshev polynomial expansion we can obtain a more accurate time integration. This is confirmed by numerical examples where the REM was combined with the Fourier pseudo spectral method for the spatial derivatives. We conclude that the REM for the time stepping combined with pseudo spectral operators for the spatial derivatives can be used to obtain numerically stable results with less computational effort than a conventional finite difference time stepping approach for the same level of accuracy. This suggests that this approach is useful primarily when highly accurate results are required.

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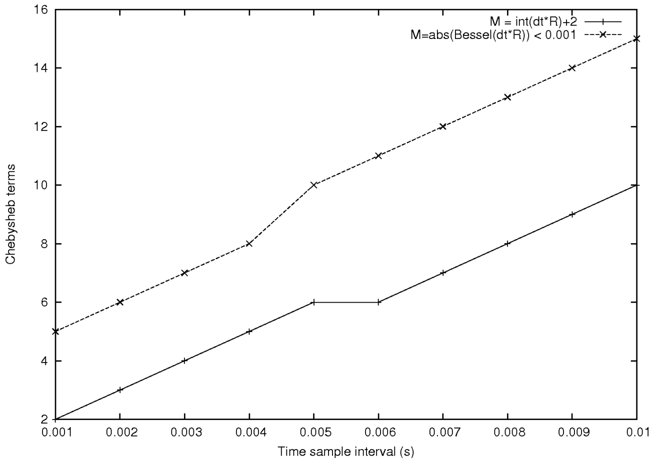
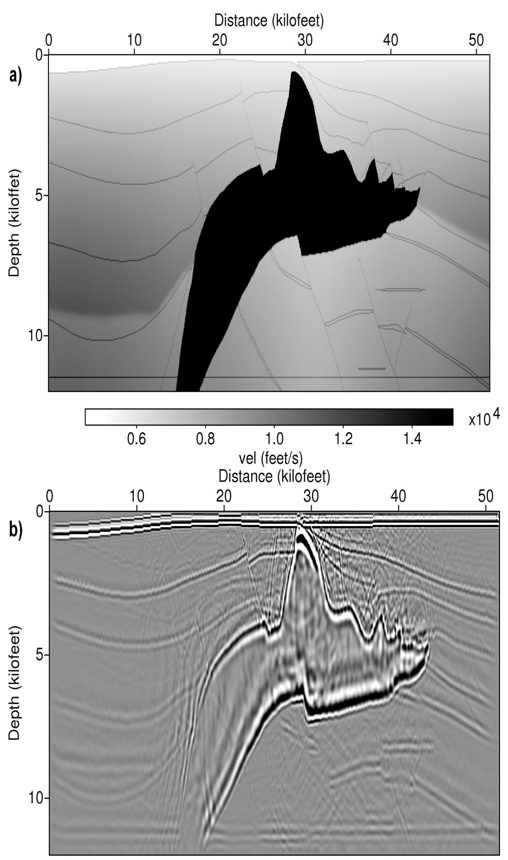


Figure 1: Plot of numbers Chebyshev polynomials required by each time step by REM versus time sample data interval.



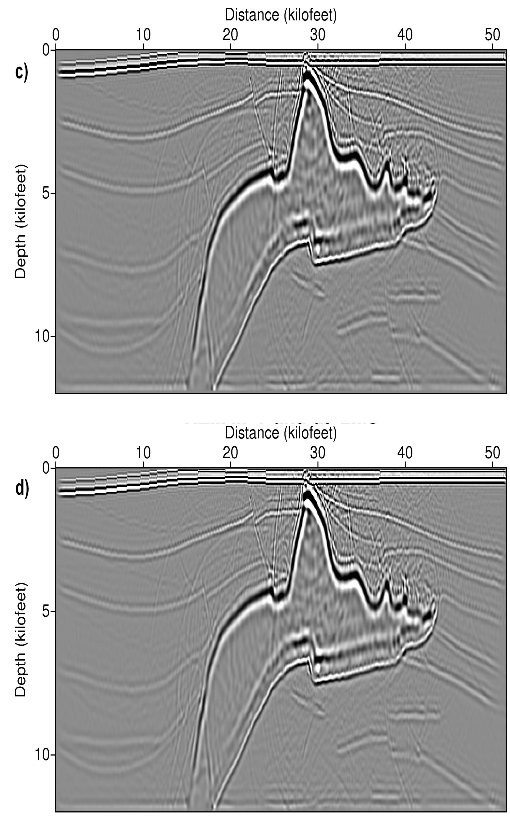


Figure 2: Velocity model (a); RTM pos-stack results using the resample data to 2ms: REM with 3 terms (b), with 6 terms (c) and the RTM result with REM using the original data (8 ms), but with 8 terms (d)

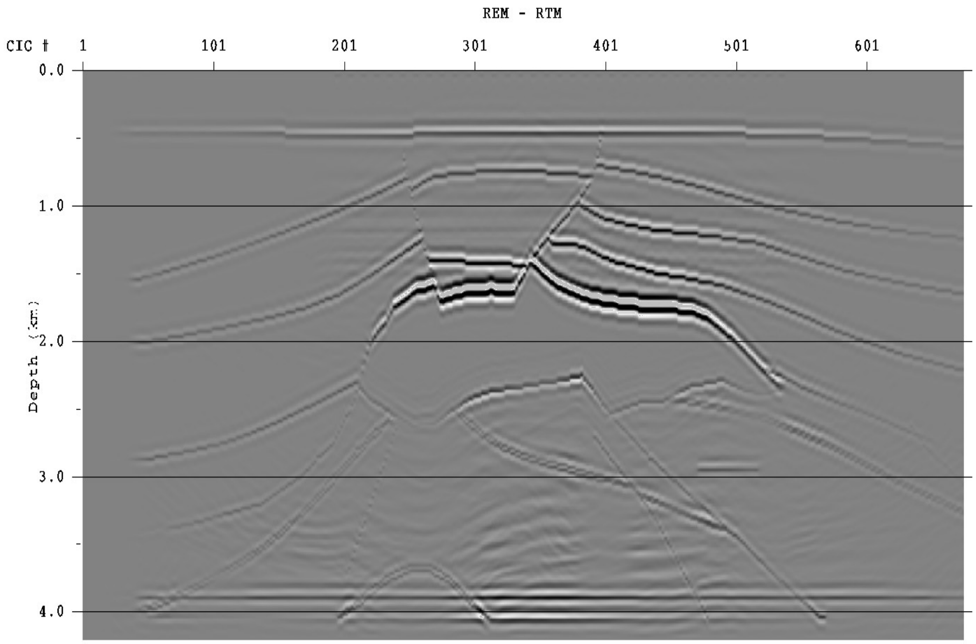


Figure 3: Pre-stack reverse time migration (RTM) by REM with time stepping obtained with 16 terms.

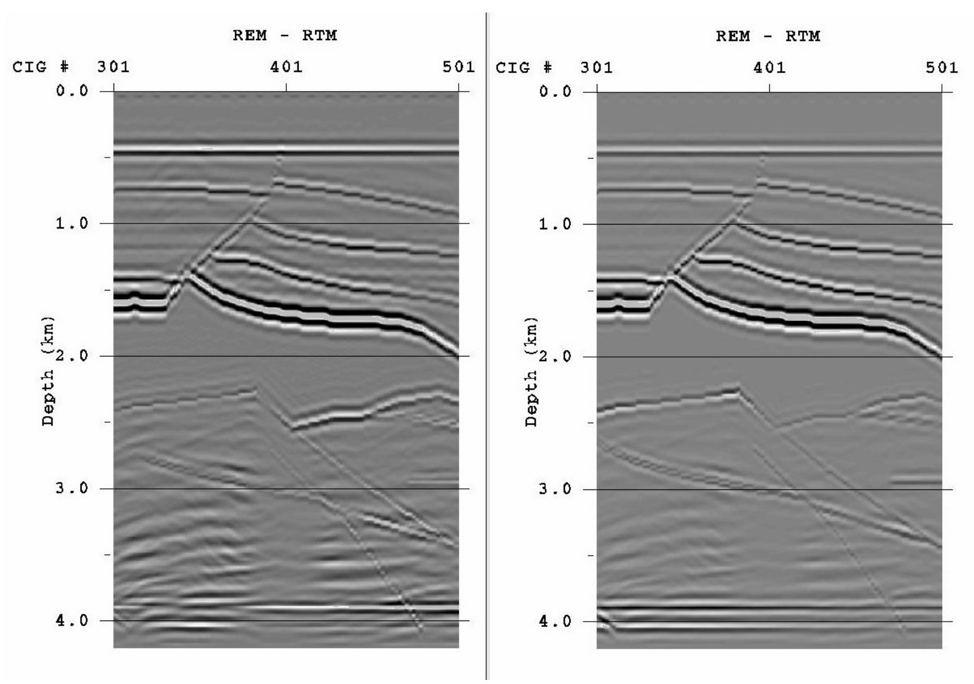


Figure 4: RTM result: Zoom of the sub-salt imaging for the REM with 8 terms (left) and the REM with 16 terms (right)

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