Wave Propagation Using Wavelet-Based Finite Elements

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Abstract

The present work discusses an alternative to the usual finite difference (FDM) approach to 2-D wave equation modeling. This approach is based on the Finite Element Method (FEM) and introduces Deslauriers-Dubuc wavelets (Interpolets) as interpolating functions. An example in 1-D was formulated using Central Difference and Newmark schemes for time differentiation. Encouraging results were obtained even for large time steps. Results obtained in 2-D with FEM and FDM are compared for validation.

Introduction

Among the numerous techniques available for the solution of the partial differential equation that describes wave propagation, the finite difference approach (Kelly et al, 1976) is by far the most employed one, being used frequently as a standard for the validation of new methods. As a disadvantage, the FDM is known for requiring excessive refining of the model discretization. Irregular grids can be used (Opršal and Zahradník, 1999), increasing the complexity of the implementation and computational cost.

The Finite Element Method (FEM) is a versatile tool for solving numerical problems. Its main advantage over other methods is its geometrical flexibility, which allows the use of complex and irregular meshes. Another known advantage of the FEM is how it naturally deals with boundary conditions. Its application to wave propagation problems is still limited, due to the large number of frequencies that are excited in the system (Bathe and Wilson, 1976). This work proposes adaptations in the FEM which can improve its performance in wave propagation problems.

The conventional formulation of the FEM uses polynomials for interpolating the displacement within the elements (shape functions). This work proposes the use of wavelets as shape functions in order to obtain satisfactory results with less refined meshes and bigger time steps than the traditional FEM and FDM would require without affecting stability and convergence.

Wavelets have several properties that are quite useful for representing solutions of partial differential equations (PDEs), such as orthogonality, compact support and exact representation of polynomials of a certain degree. These characteristics allow the efficient and stable calculation of functions with high gradients or singularities at different levels of resolution (Qian and Weiss, 1992).

A complete basis of wavelets can be generated through dilation and translation of a mother scaling function. Although many applications use only the wavelet filter coefficients of the multiresolution analysis, there are some which explicitly require the values of the basis functions and their derivatives, such as the Wavelet Finite Element Method (WFEM).

Compactly supported wavelets have a finite number of derivatives which can be highly oscillatory. This makes the numerical evaluation of integrals of their inner products difficult and unstable. Those integrals are called connection coefficients and they appear naturally in a Finite Element scheme. Due to some properties of wavelet functions, these coefficients can be obtained by solving an eigenvalue problem using filter coefficients.

Working with dyadically refined grids, Deslauriers and Dubuc (1989) obtained a new family of wavelets with interpolating properties, later called *Interpolets*. Their filter coefficients are obtained from the autocorrelation of the Daubechies' coefficients. In consequence, interpolets are symmetric, which is especially interesting in numerical analysis. The use of interpolets instead of Daubechies' wavelets considerably improves the elements accuracy.

The use of wavelets as interpolating functions in finite element formulation holds some promise due to their compact support, localization and multi-resolution properties. The approximation of the solution can be improved by increasing either the mesh resolution or the order of the wavelet used.

The formulation of an interpolet-based finite element system is demonstrated for a one-dimensional wave propagation problem. Newmark's algorithm for direct integration was tried as an alternative to the Central Difference Method in order to allow bigger time steps. A homogeneous example was formulated in order to validate the interpolet approach in 2-D problems.

Interpolets

Multi-resolution analysis using orthogonal, compactly supported wavelets has become increasingly popular in numerical simulation. Wavelets are localized in space, which allows the analysis of local variations of the problem at various levels of resolution.

In the following expression, known as the two-scale relation, a_k are the filter coefficients of the wavelet scale function.

$$\varphi(x) = \sum_{k=1-N}^{N-1} a_k \varphi(2x-k) = \sum_{k=1-N}^{N-1} a_k \varphi_k(2x)$$

The basic characteristics of interpolating wavelets require that the mother scaling function satisfies the following condition (Shi et al, 1999):

$$\varphi(k) = \delta_{0,k} = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \quad k \in \mathbb{Z}$$

The Deslauriers-Dubuc (1989) interpolating function of order N is given by an autocorrelation of the Daubechies' scaling filter coefficients (h_m) of the same order (i.e. $N/2$ vanishing moments). Its support is given by [1-N,N-1], it has even symmetry and is capable of representing polynomials of order up to N-1.

$$a_k = \sum_{m=0}^{N-1} h_m h_{m-k}$$

Interpolets satisfy the same requirements as other wavelets, specially the two-scale relation, which is fundamental for their use as interpolating functions in a FEM model. Figure 1 shows the interpolet IN6 (autocorrelation of DB6). Its symmetry and interpolating properties are evident. There is only one integer abscissa which evaluates to a non-zero value.

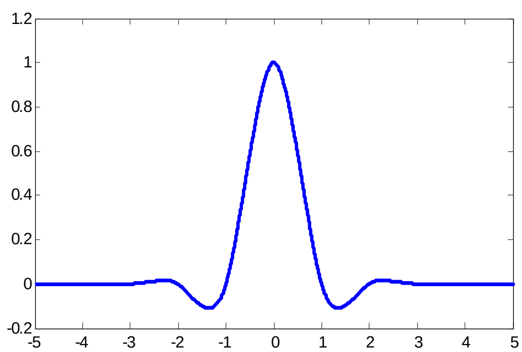

Figure 1: Interpolet IN6 scaling function with its full support.

Wave Propagation by FEM Dynamic Analysis

The partial differential equation (PDE) which rules the wave propagation is:

$$(\lambda + 2\mu)\frac{\partial^2 u(x,t)}{\partial x^2} = \rho \frac{\partial^2 u(x,t)}{\partial t^2}$$

where, u is the horizontal displacement, ρ is the density, t is the time and λ and μ are the Lamé parameters of the medium. Applying Hamilton's Principle (Clough e Penzien, 1975) and using the FEM, the PDE can be rewritten at a specific time t as a system of linear equations, which in matrix form is:

$$\mathbf{M}\left.\frac{\partial^2 \mathbf{u}}{\partial t^2}\right|_t + \mathbf{K}\,{}^t\mathbf{u} = \mathbf{0}$$

In the given expression, **M** represents the mass matrix and **K** is the stiffness matrix of the model. These FEM system global matrices are assembled from the individual contributions of the local matrices of each element.

As in the FDM, it becomes necessary to solve the system of equations at discrete time intervals. There are several effective direct integration methods, among which the most intuitive one is the Central Difference Method:

$$\left.\frac{\partial^2 \mathbf{u}}{\partial t^2}\right|_t \cong \frac{{}^{t+\Delta t}\mathbf{u} - 2\,{}^t\mathbf{u} + {}^{t-\Delta t}\mathbf{u}}{(\Delta t)^2}$$

Substituting the expression of the acceleration obtained by the Central Difference Method and solving for the next time step ${}^{t+\Delta t}\mathbf{u}$:

$${}^{t+\Delta t}\mathbf{u} = 2\,{}^t\mathbf{u} - {}^{t-\Delta t}\mathbf{u} - (\Delta t)^2\,\mathbf{M}^{-1}\mathbf{K}\,{}^t\mathbf{u}$$

The result of the matrix operation $\mathbf{M}^{-1}\mathbf{K}$ is easily obtained if **M** is diagonal. In this case, one can generate a unique local matrix at element level which contains both mass and stiffness information. It is known that the diagonal mass matrix can be used instead of the one generated from shape functions (known as consistent matrix) without introducing significant errors in the system (Burgos et al, 2007).

Nevertheless, consistent mass matrices were used in this work and a global matrix **X** was assembled using mass and stiffness contributions to the system.

Stability of the Central Difference Method is conditioned to the choice of the time step, whose upper bound is obtained from a generalized eigenvalue problem.

$$(\mathbf{K} - \omega^2\,\mathbf{M})\mathbf{u} = \mathbf{0} \rightarrow (\mathbf{X} - \omega^2\,\mathbf{I})\mathbf{u} = \mathbf{0}$$

$$\Delta t_{\max} = \frac{\sqrt{2}}{\omega_{\max}}$$

The Newmark Method

The constant-average-acceleration method (a special case of the Newmark Method) consists in an unconditionally stable scheme which can be summarized by the following expression:

$$\left(\frac{4}{\Delta t^2}\mathbf{I} + \mathbf{X}\right){}^{t+\Delta t}\mathbf{u} = \frac{4}{\Delta t^2}\,{}^t\mathbf{u} - \frac{4}{\Delta t}\,{}^t\dot{\mathbf{u}} + \frac{{}^t\ddot{\mathbf{u}}}{4}$$

This method requires the calculation of velocity and acceleration at all time steps and the inversion of the left hand side operator. Despite this, the Newmark Method is unconditionally stable. Therefore, significantly bigger time steps can be used.

Application to 2-D Problems

The PDE for the 2-D axial displacement wave equation is:

$$\frac{\partial^2 u(x,z,t)}{\partial t^2} = \frac{\lambda + 2\mu}{\rho}\left(\frac{\partial^2 u(x,z,t)}{\partial x^2} + \frac{\partial^2 u(x,z,t)}{\partial z^2}\right)$$

which can still be solved using 1-D finite elements by applying a scheme similar to what is commonly done in the FDM.

$${}^{t+\Delta t}\mathbf{u} = 2\,{}^t\mathbf{u} - {}^{t-\Delta t}\mathbf{u} - (\Delta t)^2\left(\mathbf{X}\,{}^t\mathbf{u} - {}^t\mathbf{u}\,\mathbf{Z}^\mathsf{T}\right)$$

In the expression above, a second stiffness matrix is introduced as a differential operator in the second space dimension. The displacement is also represented as a matrix, instead of the usual vector representation of the FEM.

Matrices **X** e $\mathbf{Z}^\mathbf{T}$ perform the differentiation along x and z directions respectively. These matrices contain both mass and stiffness information. Assuming spatial sampling of N_x x N_z points, **X** and **Z** are of size N_x x N_x and N_z x N_z, respectively. This represents an improvement over traditional FEM in terms of computational effort.

Usually, the FEM requires a displacement vector of length N_xN_z and both global stiffness and mass matrices must have a size of N_xN_z x N_xN_z, operating over the whole system.

The application of the Newmark Method in this 2-D scheme is not as simple as in 1-D, since the elements are still one-dimensional and there are right and left matrix multiplications involved. The following expression is an adaptation of the Newmark Method to the 2-D implementation proposed.

$$\left(\frac{4}{\Delta t^2}\mathbf{I}+\mathbf{X}\right){}^{t+\Delta t}\mathbf{u} = \frac{4}{\Delta t^2}\,{}^{t}\mathbf{u} - \frac{4}{\Delta t}\,{}^{t}\dot{\mathbf{u}} + \frac{{}^{t}\ddot{\mathbf{u}}}{4} - {}^{t+\Delta t}\mathbf{u}\,\mathbf{Z}^\mathbf{T}$$

In this expression, ${}^{t+\Delta t}\mathbf{u}$ appears in both sides of the equation, thus requiring the application of an iterative method for its calculation. This procedure not only increases the computational effort significantly but can also cause instability for large time steps.

Element Formulation

Assuming that displacement u is approximated by a series of interpolating scale functions, the following may be written:

$$u(\xi) = \sum_{k=2-N}^{N-1} \alpha_k \varphi(\xi - k)$$

Stiffness and mass matrices can be obtained by solving the wave propagation PDE using the FEM. Adimensional coordinates (ξ) within the interval [0,1] are used in wavelet space, which leads to the subsequent expressions:

$$\bar{k}_{i,j} = (\lambda + 2\mu)\int_0^1 \varphi_i'(\xi)\varphi_j'(\xi)d\xi = (\lambda+2\mu)\,\Lambda_{i,j}^{1,1}$$

$$\bar{m}_{i,j} = \rho\int_0^1 \varphi_i(\xi)\varphi_j(\xi)d\xi = \rho\,\Lambda_{i,j}^{0,0}$$

The so-called connection coefficients Λ appear in the expressions above. Wavelet dilation and translation properties allow the calculation of connection coefficients to be summarized by the solution of an eigenvalue problem based only on filter coefficients (Zhou & Zhang, 1998).

$$\left(\mathbf{P} - \frac{1}{2^{d_1+d_2-1}}\mathbf{I}\right)\mathbf{\Lambda}^{\mathbf{d_1,d_2}} = \mathbf{0}$$

$$\mathbf{P} = \left[a_{r-2i}a_{s-2j} + a_{r-2i+1}a_{s-2j+1}\right]_{i,j,r,s=(2-N)\ldots(N-1)}$$

Since the expression above leads to an infinite number of solutions, there is the need for a normalization rule that provides a unique eigenvector. This unique solution comes with the inclusion of the so-called moment equation, derived from the wavelet property of exact polynomial representation (Latto et al, 1992).

$$\sum_i\sum_j M_i^k M_j^k \Lambda_{i,j}^{d_1,d_2} = \frac{(k!)^2}{(k-d_1)!(k-d_2)!(2k-d_1-d_2+1)}$$

$$M_i^j = \frac{1}{2^{j+1}-2}\sum_{k=0}^{j}\binom{j}{k} i^{j-k}\sum_{l=0}^{k-1}\binom{k}{l}M_0^l\left(\sum_{i=0}^{N-1}a_i\, i^{k-l}\right)$$

The adimensional expressions for the stiffness $\bar{\mathbf{k}}$ and mass $\bar{\mathbf{m}}$ matrices are in wavelet space and need to be transformed to physical space, using a transformation matrix **T** obtained by evaluating the wavelet basis at the element node coordinates.

$$\mathbf{T} = \left[\varphi\left(\frac{j-1}{2N-3} + N - (i+1)\right)\right]_{i,j=1..(2N-2)}$$

$$\mathbf{k} = \frac{1}{L}\left(\mathbf{T}^{-T}\,\bar{\mathbf{k}}\,\mathbf{T}^{-1}\right)$$

$$\mathbf{m} = L\left(\mathbf{T}^{-T}\,\bar{\mathbf{m}}\,\mathbf{T}^{-1}\right)$$

It can be noticed that some terms related to the length of the element emerge from coordinate changes.

Examples

To validate the use of the Newmark Method for the Wavelet Finite Element, a 1-D example was formulated. It consists in applying a forced displacement at the free end of a pinned, unit length rod. The propagation was modeled by the FDM using 601 points and Δt=0.1*ms*. For the FEM based on the IN6 interpolet, the discretization was performed with 91 points and Δt=1*ms.* The rod's middle point time response for both methods is shown in figure 2, which shows that the FEM result is acceptably close to that of the FDM, especially considering that FEM generates over 60 times less data per simulated second.

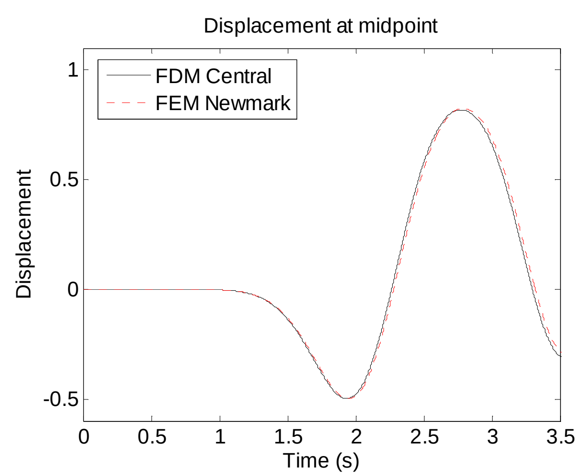

Figure 2: FEM results using IN6 Interpolet and Newmark Method (Δt=1*ms*) compared to FDM (Δt=0.1*ms*) for a one-dimensional wave propagation.

In order to validate the 2-D formulation, a simple example was proposed by analyzing the propagation in a 2*km* x 2*km* model with constant velocity α=3000*m*/*s*.

Finite Element model was sampled every 6.17*m* in both directions with a time step Δt of 0.46*ms*, obtained by the generalized eigenvalue problem already described. Non-integer values appear due to the 11 degrees-of-freedom present in the element implemented with IN6 (order 6 Interpolet). A Newmark scheme was also implemented using the same spatial sampling and Δt=0.6*ms*. The results were compared to the FDM using 5*m* spacing in both directions and Δt=0.2*ms*. Finite Element mesh has 325 x 325 points and Finite Difference mesh has 401 x 401. Results are shown in figure 3.

Even with a less refined mesh and a more than twice bigger time step, the IN6 element was able to identify characteristic peaks of the wave propagation. The results of the proposed adaptation to the Newmark Method were very similar to the Central Difference ones using an even bigger time step.

Conclusions

This work presented the formulation and validation of an interpolet-based finite element. Newmark's method for time discretization appears promising, although its application in 2-D problems with 1-D elements remains a challenge. The main improvement in the presented formulation was the possibility of using a bigger time step than the one required by the FDM. In future works, models with greater complexity will be analyzed and different families of wavelets will be explored.

As done in the traditional FEM, all matrices involved can be stored and operated in a sparse form, since most of their components are null, thus saving computer resources.

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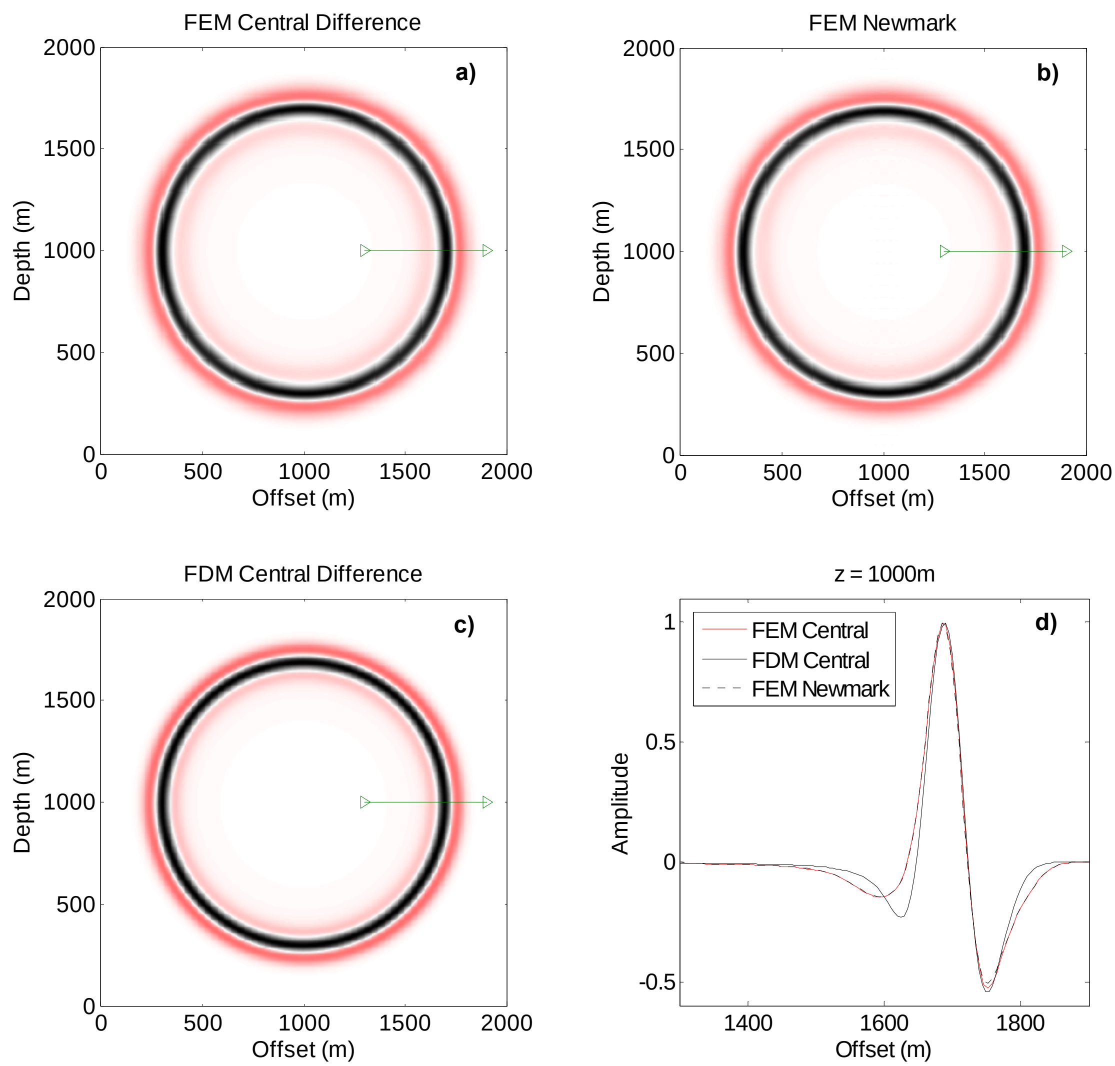

Figure 3: Propagation snapshots at time t=0.3*s* for Example 2 using: (a) FEM and Central Difference Method, (b) FEM and Newmark Method, (c) FDM and Central Difference Method; (d) comparison of amplitudes along the indicated segment at depth 1000*m.*

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