Low-frequency extension of the Backus averaging method

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Abstract

The standard Backus (1962) averaging method is widely used for upscaling of the well-log data to seismic frequency range. In many cases, with strong heterogeneity within the upscaling unit, the Backus averaging is not accurate enough. We propose to extend the Backus averaging method for the low-frequency case and introduce the dispersive Backus model using the Baker-Campbell-Hausdorff (BCH) series (Serre, 1965). We derive the first- and second-order terms of this series, and extend this technique to the medium with arbitrary number of layers in a period. That results in the correction term for velocity dispersion at low frequencies. We show that the phase velocity in such media is the even function of frequency.

The accurate description of velocity dispersion for effective medium is very important in seismic modelling and inversion of seismic data into effective reservoir properties.

Introduction

At low frequencies, when the wavelength is much larger than the period of the stack of layers, the layered medium has the properties of effective homogeneous anisotropic medium (Rytov, 1956; Backus, 1962; Schoenberg and Muir, 1989). When frequency increases, we observe the velocity dispersion (Helbig, 1984; Norris, 1992) and periodically located pass- and stop-bands with propagation and attenuation of the waves (Stovas and Arntsen, 2006; Stovas and Ursin, 2007; Roganov and Roganov, 2008). That leads to the specific filtering of the waves (Braga and Herrmann, 1992) and results in the frequency-dependent caustics in the group domain (Roganov and Stovas, 2011).

In this paper, we derive the correction terms that control the velocity dispersion at low frequencies. Similar equations can be found in Santosa and Symes (1991) and Norris (1992) for different type of media based on the analysis of the asymptotic behavior of the roots of the Floquet's equation given in terms of propagator $\mathbf{P}(\omega)$ for a periodically layered medium (Gilbert and Backus, 1966; Kennett, 1983; Nayfeh, 1989; Rousseau, 1989). The derived approximation for the phase velocity is more accurate than the one proposed by Stovas (2007) for the weak-contrast finely layered medium.

In order to approximate of the right-side part of the system of differential equations for given frequency and horizontal slowness we use the logarithm of propagator matrix $\tilde{\mathbf{M}}(\omega)=\frac{1}{i\omega H}\log\mathbf{P}(\omega)$. This matrix is correctly defined for a given frequency ω if the matrix $\mathbf{P}(\omega)$ can be transformed to the diagonal matrix with different eigen-values on the main diagonal. If some of these eigen-values are equal, the correct definition of the logarithm requires the condition that the corresponding eigen-values of matrix $\tilde{\mathbf{M}}(\omega)$ to be located at the same Riemann surface. The last condition is satisfied if all Bloch's waves are in the first pass-band or the Brillouin zone (Brillouin, 1953; Shuvalov, Kutsenko and Norris, 2010).

Taking into account that matrix $\tilde{\mathbf{M}}(\omega)$ is given by the logarithm from the product of the exponent matrices; we can apply the Baker-Campbell-Hausdorff (BCH) series (Baker, 1898; Campbell, 1897; Hausdorff, 1927; Serre, 1965). In mathematics, the BCH series is used to construct the Lie groups for the Lie algebras (Bourbaki, 1971). Using this technique, we derive the expressions for the terms of Taylor series with respect to ω for matrix $\tilde{\mathbf{M}}(\omega)$. The zero-order term in this series gives the well-known Backus averaging (Backus, 1962).

We derive the first- and second-order terms of this series, and extend this technique to the medium with arbitrary number of layers in a period. That results in the correction term for velocity dispersion at low frequencies. We show that the dispersion equation in such media is the even function of frequency.

The theory we develop in this paper can be used for an extension of the Backus averaging technique for the low frequency wave propagation. This is an important issue for matching of well-log data with seismic data, smoothing of the well-log data, upscaling of the well-log data and seismic modeling.

The wave propagation in finely layered media

The vector containing the stress-strain components $\mathbf{f}_j(z)$ defined for each layer, satisfies the differential equation,

$$\frac{d\mathbf{f}(z)}{dz}=i\omega\mathbf{M}_j(\omega)\mathbf{f}(z), \tag{1}$$

where $i=\sqrt{-1}$, ω is the frequency and matrix $\mathbf{M}_j(\omega)$ is defined by the horizontal slowness and the type of the medium. We can define the matrix $\tilde{\mathbf{M}}(\omega)$ by following equation

$$\tilde{\mathbf{M}}(\omega)=\frac{1}{i\omega H}\log\mathbf{P}(\omega)=\mathbf{E}^{-1}\operatorname{diag}(q_m)\mathbf{E}, \tag{2}$$

where the matrix $\mathbf{E}$ is composed from the eigen-vectors of matrix $\mathbf{P}(\omega)$ and $H=\sum_{j=1}^{N} z_j$ is the overall layer thickness (reservoir thickness). Let us denote $\mathbf{A}_k=\sum_{j=1}^{N} z_j^k \mathbf{M}_j^k$. The BCH series is given by

$$H\tilde{\mathbf{M}}(\omega)=\tilde{\mathbf{M}}_0+i\omega\tilde{\mathbf{M}}_1-\omega^2\tilde{\mathbf{M}}_2+o(\omega^3), \quad (3)$$

$$\tilde{\mathbf{M}}_0=\mathbf{A}_1, \quad \tilde{\mathbf{M}}_1=\frac{1}{2}\sum_{N\ge i>j\ge 1} z_i z_j\left(\mathbf{M}_i\mathbf{M}_j-\mathbf{M}_j\mathbf{M}_i\right),$$

$$\tilde{\mathbf{M}}_2=-\frac{1}{6}\mathbf{A}_1^3+\frac{1}{4}\left(\mathbf{A}_1\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1\right)-\frac{1}{3}\mathbf{A}_3 +\frac{1}{2}\sum_{m\ge i>j>k\ge 1} z_i z_j z_k\left(\mathbf{M}_i\mathbf{M}_j\mathbf{M}_k+\mathbf{M}_k\mathbf{M}_j\mathbf{M}_i\right). \quad (4)$$

For the vertical propagation and finite number of isotropic or transversely isotropic layers, the elements of matrices $\tilde{\mathbf{M}}_0$, $\tilde{\mathbf{M}}_1$ and $\tilde{\mathbf{M}}_2$ can be given by simple equations. Note that matrix $\tilde{\mathbf{M}}_0$ corresponds to the standard effective Backus (1962) medium. For a single layer *j,* we have

$$z_j\mathbf{M}_j=t_j\begin{pmatrix}0 & Z_j^{-1}\\ Z_j & 0\end{pmatrix}, \quad (5)$$

where $Z_j=\rho_j V_j$ is the impedance, $t_j=z_j/V_j$ is the vertical traveltime in layer *j*, ρ_j is the density and V_j is the vertical velocity. From equations (4) we obtain

$$\tilde{\mathbf{M}}_0=\begin{pmatrix}0 & b_0\\ a_0 & 0\end{pmatrix}, \quad \tilde{\mathbf{M}}_1=\begin{pmatrix}-a_1 & 0\\ 0 & a_1\end{pmatrix},$$

$$\tilde{\mathbf{M}}_2=\begin{pmatrix}0 & b_2\\ a_2 & 0\end{pmatrix}, \quad (6)$$

with

$$a_0=\sum_{j=1}^{N} t_j Z_j, \quad b_0=\sum_{j=1}^{N}\frac{t_j}{Z_j},$$

$$a_1=\frac{1}{2}\sum_{N\ge i>j\ge 1} t_i t_j\left(\frac{Z_i}{Z_j}-\frac{Z_j}{Z_i}\right), \quad (7)$$

and

$$a_2=-\frac{1}{6}a_1^2 b_1+\frac{1}{2}a_1\sum_{j=1}^{N} t_j^2-\frac{1}{3}\sum_{j=1}^{N} t_j^3 Z_j +\sum_{N\ge i_3>i_2>i_1\ge 1} t_{i_1}t_{i_2}t_{i_3}\frac{Z_{i_1}Z_{i_3}}{Z_{i_2}}$$

$$b_2=-\frac{1}{6}a_1 b_1^2+\frac{1}{2}b_1\sum_{j=1}^{N} t_j^2-\frac{1}{3}\sum_{j=1}^{N}\frac{t_j^3}{Z_j} +\sum_{N\ge i_3>i_2>i_1\ge 1} t_{i_1}t_{i_2}t_{i_3}\frac{Z_{i_2}}{Z_{i_1}Z_{i_3}}, \quad (8)$$

The wave propagation in a finely layered medium was investigated in many papers (see, for example, Stovas and Arntsen (2006) and Stovas (2007)).

Periodically layered medium

The periodically layered medium is an important tool to analyze the vertically heterogeneous reservoirs consisting of shale and sand layers. For quasi-vertical propagation, the important parameter is the reflection coefficient computed at single shale-sand interface. The implicit formulas for reflection and transmission responses in a periodically layered medium are derived in Stovas and Ursin (2007). An example of reservoir model and synthetic seismic computed by assuming periodically layered sand-shale sequence within reservoir body is shown in Figure 1.

The dispersion equation for a homogeneous reservoir can be defined in terms of reflection coefficient $r=(Z_2-Z_1)/(Z_2+Z_1)$ at interface between the layers,

$$\frac{1}{V^2(\omega)}=\left(\frac{\alpha_1}{V_1}+\frac{\alpha_2}{V_2}\right)^2+\frac{4r^2}{1-r^2}\frac{\alpha_2\alpha_1}{V_2V_1} +\omega^2H^2\frac{4\alpha_1^2\alpha_2^2r^2}{3V_1^2V_2^2\left(1-r^2\right)^2} +\omega^4H^4\frac{4\alpha_1^2\alpha_2^2r^2}{45V_1^2V_2^2\left(1-r^2\right)^2}\left[\left(\frac{\alpha_1}{V_1}+\frac{\alpha_2}{V_2}\right)^2+\frac{2\alpha_1\alpha_2\left(1+3r^2\right)}{V_1V_2\left(1-r^2\right)}\right], \quad (9)$$

where $Z_j=\rho_j V_j,\ j=1,2$, are the elastic impedances from shale and sand layers and $\alpha_j,\ j=1,2$ are the fractions for shale and sand.

The first term in (9) corresponds to the time-average (ray) velocity that is the infinite-frequency limit. The sum of the first and the second terms represents the Backus (1962) velocity that is the zero-frequency limit. The terms at ω^2 and ω^4 are the first and the second correction terms, respectively.

Examples

In order to illustrate the velocity dispersion equation, we consider the periodically layered model consisting of two layers with the following properties, $V_{P1} = 2000\, m/s$, $V_{S1} = 700\, m/s$, $\rho_1 = 2000\, kg/m^3$, $V_{P2} = 3000\, m/s$, $V_{S2} = 900\, m/s$ and $\rho_2 = 2200\, kg/m^3$. In Figure 1, one can see the angle-dependent velocity dispersion contour plot (top-left) indicating the frequency-dependent induced anisotropy. The effective phase velocity, the Backus velocity and the Backus velocity with correction versus horizontal slowness for 2-layer velocity model (top right). The effective phase velocity versus horizontal slowness for 2-layer velocity model computed for frequency 10Hz, 20Hz, 30Hz, 40Hz and 50Hz (bottom left). The Backus and the ray velocities are shown by red and blue solid lines, respectively. Velocity versus frequency taken at different values of horizontal slowness is shown in Figure 1 (bottom right). One can see that the dispersive Backus velocity model results in more accurate velocity description at low frequencies for the finely layered medium. In Figure 2 one can see the results of the smoothing the well-log data (left) and errors in effective velocity by using the Backus averaging operator and corrected Backus operator computed for frequencies of 10 and 30 Hz. The errors with applying the proposed method are much smaller then those computed by standard Backus method. In Figure 3 (top), one can see the time-average velocity, the Backus velocity and the dispersive Backus velocity with the second- and fourth-order terms from equation (9) by considering the shale-sand sequence with parameters mentioned above. The error in effective velocity taken at frequency of 20Hz is about 60m/s. This error can result in uncertainties for net-to-gross estimation or fluid saturation estimation if seismic data are recorded with central frequency of 20 Hz (Figure 3, bottom).

Conclusions

To compute the effective matrix from the stack of the layers we use the BCH series. From the truncated BCH series we derive the velocity dispersion equation that correctly describes the wave propagation at low frequencies. The explicit equations derived for an acoustic medium, periodically layered medium, medium with monoclinic anisotropy and the vertical propagation case.

The derived equations are tested on the two-layer periodically layered medium and on the real well-log data. The first-order correction term in the velocity dispersion equation results in more accurate phase velocity at low frequencies. This correction is an extension of the Backus averaging method and can be used for upscaling of the well-log data and seismic modeling.

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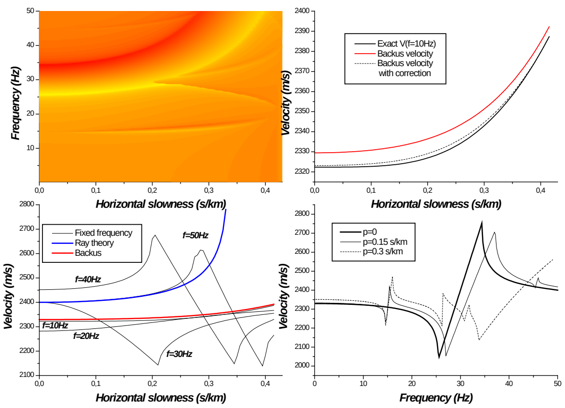

Figure 1. The effective P-wave phase velocity versus horizontal slowness and frequency for 2-layer velocity model (top left). The effective phase velocity, the Backus velocity and the Backus velocity with correction versus horizontal slowness for 2-layer velocity model (top right). The effective phase velocity versus horizontal slowness for 2-layer velocity model computed for frequency 10Hz, 20Hz, 30Hz, 40Hz and 50Hz (bottom left). The Backus and the ray velocities are shown by red and blue solid lines, respectively. Velocity versus frequency taken at different values of horizontal slowness (bottom right).

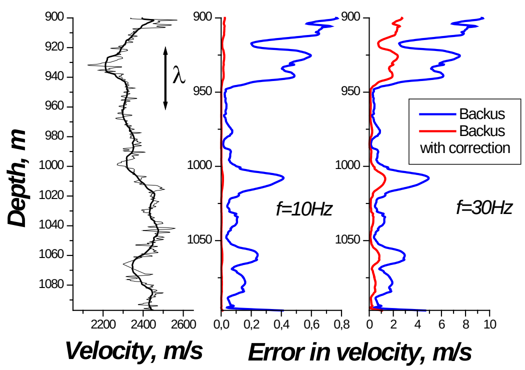

Figure 2. Smoothing of the vertical P-wave velocity from the well-log data (left) and errors in the Backus velocity shown by blue line and the Backus velocity with correction shown by red line for frequencies of 10 and 30 Hz.

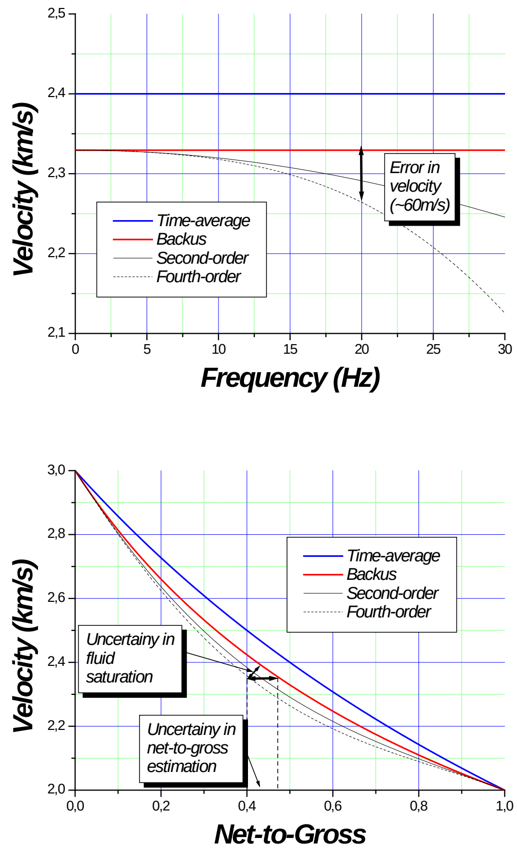

Figure 3. The blocking model from the well-log data. The P-wave velocity profile is shown to the top and the exact, the Backus and the Backus with correction velocities versus frequencies are shown by black, red and dashed line, respectively.