Finite difference modelling of galvanic current flow in inhomogeneous conductive media

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Abstract

This is a relatively simple galvanic current flow boundary value problem to demonstrate that the Finite Difference Frequency Domain method can accurately model current flow through a bed boundary. The method uses non-uniform grids to be able to model rapidly changing fields. The formulation uses secondary potentials to remove singular problems with primary fields.

Introduction

The representation of fields and waves in physical problems is usually done by partial differential equations. A solution is found when the field or wave, governed by the partial differential equation, satisfies the particular set of boundary conditions which are chosen according to the given physical situation.

In most of geophysical problems, due the complex geometry of the boundary value problem, it is impossible to obtain an analytical solution, so we need to use numerical procedures to obtain approximate solutions. The most commonly used techiques to accomplish these numerical tasks are the finite difference (FD) and finite element (FE) method (Schenkel, 1991). Both methods require a complete gridding of the solution domain, which must be fine enough to track the features of interest, usually the inhomogeneities.

Here we use the finite difference frequency domain method to model a Schlumberger resistivity sounding in an environment involving two infinite half spaces. The non-uniform grid is refined in the interface between the two media, where we have a rapidly variation of the electrical field.

The Schlumberger array consists of two source electrodes that are inserted into the ground providing a flow of current through the earth from one electrode to the other and two receiver electrodes located betweeen the source electrodes, to measure the voltage difference. The flow lines are always perpendicular to the equipotentials surfaces, so in our case, where we have the media interface parallel and next to the line of electrodes, the current flows through the interface.

This work is an initial example to test some issues needed in future problems. Our goal is to assess electromagnetic problems in more complex 2 1/2-D and 3-D environments, like marine oil soundings, with irregular seabed and/or anisotropy. For these problems, the matrices that arise from the use of finite difference method become large, requiring significant computational power. Non-uniform grids and exploitation of the sparse property of the finite difference matrix are example options to circumvent this problem.

Further, the use of *gradient methods* to solve the system of equation is tested here. We started using the conjugate gradient method (CG) that takes advantage of the complex symmetry of the FD matrices of this problem to make the calculations faster. But when handling anisotropy and bathymetry, which are our next steps, this complex symmetry is lost, requiring the biconjugate gradient method (BiCG), which does't require this symmetry but is slower than CG.

Matrix *preconditioning* can improve the search for the solution, but it can also destroy the complex symmetry of the matrices and this is another reason for the use of BiCG. There is much literature and research on matrix preconditioning and iterative methods, for example: Chen (2005), Barret et al. (1994), Greenbaum (1997) and Saad (1996).

Planned extensions are the use of more general BiCG methods that allow the implementation of LU incomplete preconditioning to significantly reduce the number of iterations per wave number.

Method

In an isotropic medium, current density $\mathbf{J}$ and electric field intensity $\mathbf{E}$ are related by Ohm's Law:

$$\mathbf{J} = \sigma \mathbf{E} \tag{1}$$

where σ is the conductivity (=1/resistivity).

The electric field is equal to the gradient of the electrical potential Φ:

$$\mathbf{E} = -\nabla \Phi \tag{2}$$

From the principle of conservartion of charge, we also have:

$$\nabla.\mathbf{J} = -\mathscr{I}\delta(x_s)\delta(y_s)\delta(z_s) \quad (3)$$

where $\mathscr{I}$ is the electric current and x_s, y_s and z_s are the cartesian coordinates of the point source (Wait, 1982). Thus, the above equation holds everywhere except at the source itself.

Combining equations (1),(2) and (3), we obtain:

$$-\nabla[\sigma(x,y,z)\nabla\Phi(x,y,z)] = \mathscr{I}\delta(x_s)\delta(y_s)\delta(z_s) \quad (4)$$

This Poisson's equation is solved for Φ. The difference in Φ at two different points is equal to the potential difference in volts.

In our problem, the conductivity varies only in z direction but Φ and $\mathscr{I}$ are x,y,z dependent. In such situations, the solution for Φ is solved more easily in the Fourier transformed space (x,k_y,z) using the equation:

$$F(x,k_y,z) = \int_{-\infty}^{+\infty} f(x,y,z)e^{-ik_y y}dy \quad (5)$$

Applying the Fourier transform in equation (4) produces:

$$-\nabla_\perp[\sigma(z)\nabla_\perp\Phi(x,k_y,z)] - k_y^2\sigma(z)\Phi(x,k_y,z) = \mathscr{I}\delta(x_s)\delta(z_s) \quad (6)$$

where $\nabla_\perp = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial z}\hat{k}$.

Using the formulation for the secondary potential, where:

$$\Phi = \Phi^p + \Phi^s$$

$$\sigma = \sigma^p + \Delta\sigma$$

We obtain

$$-\nabla_\perp[\sigma(z)\nabla_\perp\Phi^s] - k_y^2\sigma(z)\Phi^s = \nabla_\perp[\Delta\sigma\nabla_\perp\Phi^p] - k_y^2\Delta\sigma\Phi^p. \quad (7)$$

Because of the first derivative of the scalar potential Φ on the left size of the above equation, the resulting matrix will not be complex symmetric as necessary for conjugate gradient solutions. As pointed out by Allers et al.(1994), Stefănescu's transform changes the electrostatic current flow potential equation to complex symmetric form . This motivates defining a new scalar potential $V(x)$ as

$$\Phi = V(x)/\alpha(x) \quad (8)$$

where

$$\alpha(x) = (\sigma(x))^{1/2}$$

Substitution of transformation (8) into equation (7) gives, upon simplification

$$\left[\nabla_\perp^2 - \frac{\nabla_\perp^2(\alpha)}{\alpha} - k_y^2\right]V^s = \left[\frac{-\Delta\alpha(\nabla_\perp^2 - k_y^2)}{\alpha} + \frac{\nabla_\perp^2(\alpha)}{\alpha}\right]V^p \quad (9)$$

which is the equation to be used with finite difference.

Data examples

To validate the code, we compare the numerical solution with its equivalent analytical. For completeness, the point electrode potential is represented by

$$\Phi_0 = \frac{\mathscr{I}_0}{4\pi^2\sigma_0}\int_{-\infty}^{+\infty} K_0(k_y\rho)e^{ik_y y}dk_y \quad (10)$$

where σ_0 is the conductivity of the source medium and $\rho = \sqrt{x^2+y^2}$ and K_n is the modified Bessel function of second kind.

For the inhomogeneous medium, we have:

$$\Phi = \begin{cases} \Phi_0 + \Phi_s & \textit{at the source medium;} \\ \Phi_t & \textit{otherwise} \end{cases}$$

where

$$\Phi_s = k\int_{-\infty}^{+\infty} a_s(k_y)I_0(k_y\rho)e^{ik_y y}dk_y$$

$$\Phi_t = k\int_{-\infty}^{+\infty} a_t(k_y)K_0(k_y\rho)e^{ik_y y}dk_y$$

and where

$$k = \frac{\mathscr{I}_0}{4\pi^2\sigma_0}$$

$$a_s(k_y) = \frac{e^{-ik_y y}K_0K_1(1-\lambda)}{I_1K_0 + \lambda I_0K_1}$$

$$a_t(k_y) = \frac{e^{-ik_y y}}{k_y z_t(I_1K_0 + \lambda I_0K_1)}$$

In these expressions I_n is the modified Bessel function of first kind, $\lambda = \sigma_1/\sigma_0$, where σ_1 is the conductivity of the medium without the source and z_t is the z coordinate of the source (Wait,1982).

The potentials were calculated in a non-uniform grid, which has greater refinement around the coordinate $z = 0$. Thus, for a $z < 0$ we have the medium conductive of $\sigma_0 = 0.1\ S/m$ and for $z > 0$ we have the medium conductive of $\sigma_1 = 1\ S/m$. The distance of the electrode sources to the bed interface is $z_t = -28.5\ m$.

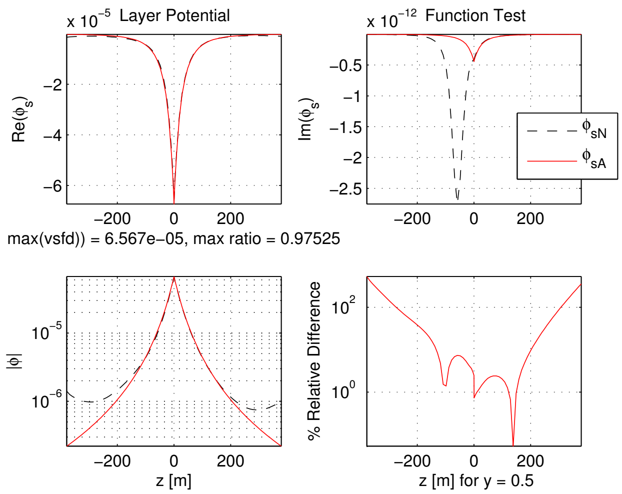

Figure 1: Comparison between the numerical (dashed) and analytical (solid) results.

Figure 1 compares the results obtained with the numerical and analytical procedures. The two graphics on top show the real and imaginary parts of the secondary potential. We can note a good match in the central region of the two solutions, that can be better verified in the two graphics on bottom. The latter two graphics show the absolute value of the secondary potential and the relative error between the analytical and numerical solutions.

The imaginary part of the solution is very small, as expected, and can be ignored. Notice that the numerical solution gets worse near the edges. It occurs because the finite difference method is less accurate in this region. The last graphic confirms that the numerical and analytical solution are very close, diverging at the edges.

In the example above, we used the conjugate gradient method to solve the system of equation with Jacobi preconditioner, which takes only the main diagonal of the matrix to be solved and multiplies it by this matrix. But, we have tested the biconjugate gradient method with a LU incomplete preconditioner and we noticed a significant reduction of iterations per wave number. This reduction was even lower when we decreased the drop tolerance of the LU incomplete preconditioner.

Summary and Conclusions

This paper shows the modelling of a direct current flow using a Schlumberger array in a medium consisting of two half spaces with different conductivities. The electric source is in one medium and next to the interface with the another one, so that there is current flowing between the media. We solved second order differential equations using finite difference and we showed that the method could deal with this current through the media interface. We noticed that the method is less efficient near the boundaries, but this isn't an issue actually, because we used a grid large enough for the decay of the electric field so we don't need to concern with reflection distortions near the boundaries. BiCG algorithms do not require specific sparse matrix symmetry. Thus we can use LU incomplete preconditioning to improve convergence. The number of iterations per wave number is then significantly decreased.

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Acknowledgments

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