Perfect Matched Layer Optimization for Acoustic Wave Modelling

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Abstract

The Perfect Matched Layer (PML) method is presented and optimized. It is shown that the implemented optimization increases the effectiveness of the absorbing layer. Furthermore, PML's effectiveness is analyzed with respect to its defining constants. Such analysis allows for faster tuning of PML to any given setup. Further results show that side effects are very sensitive to the number of grid points used in the absorbing layer, with better results found for wider layers.

Introduction

The appearance of fast processing computers and the continuous advances in numerical analysis have allowed new developments in acoustic wave modelling. Over recent years, many articles dealing with numerical simulations of wave propagation using finite difference, finite element and boundary integral methods have been published (Durran, 1999).

A problem that has been widely discussed in papers is how to express the radiation condition mathematically at a boundary which is only at a finite distance from the energy source (Sommerfeld, 1949). The boundary condition should allow travelling disturbances to pass through the limits of the domain without generating spurious reflections that propagate back toward the interior, which may eventually override the original emitted seismic signals.

A simple way to work out this problem is enlarging the computational domain, therefore delaying the backward reflections. This approach, though commonly used, requires large numerical meshes and, consequently, great CPU time. Berenger (1994) proposed the PML method for solving electromagnetic and elastic wave equations as a more efficient alternative. PML is based on designing a new matched medium intended to absorb - without reflection - the incident waves at any frequency and at any incidence angle.

This work aims at performing effectiveness tests for an optimized PML method in a finite-difference time-domain (FDTD) scheme applied to acoustic wave modelling. The objectives are to generate data that will help fine-tune PML more efficiently, obtain high absorption rates with as narrow an absorbing layer as possible, and obtain an optimized version of PML.

The PML method in its original form is presented and optimized so that wave reflection along the boundaries of the domain is reduced. It is shown that the implemented PML optimization increases the effectiveness of the absorbing layer. It is also shown that absorption is very sensitive to the number of nodes in the absorbing layer, and that wider layers yield better results. Furthermore, it is shown that PML's ideal set of constants vary in a relatively predictable way that can be used to shorten the amount of time needed to tune PML to each setup.

The Original PML Technique

Berenger (1994) introduced PML as one of the ways to abate spurious reflections on the computational domain. A set of non-physical equations is applied along the boundaries of the domain so that the energy from the wave is abated at the boundaries rather than reflected back into the domain.

If it were to be used on a continuous domain, PML would only require an infinitely narrow region along the boundaries to yield complete absorption. This is not true, however, for discrete domains such as the ones used in the finite difference method. In that case, such a region must be a given number of nodes wide, in which PML resembles other absorbing boundary methods such as the Damping Zone (Cerjan *et al.*, 1985).

The 2D continuity and linearized Euler equations take a different form in the absorption layers. Since the subject matter is acoustics (Whitham, 1999), they must be written as

$$p_t + B\alpha p = -B\nabla \cdot \vec{u}\,, \tag{1}$$

$$\vec{u}_t + B\alpha\vec{u} = -\frac{1}{\rho}\nabla p\,, \tag{2}$$

where ρ, p and $\vec{u}$ are, respectively, the medium density, the acoustic pressure and the velocity vector; α is the attenuation coefficient; B ($=\rho c^2$) is the medium bulk modulus; c is the medium wave or, for acoustic waves in particular, sound-speed.

The attenuation coefficient α is the distinguishing feature between these equations and their non-PML counterparts. It is given by the expression,

$$\alpha(i) = \frac{1}{B\delta t}\ln\left(\frac{1}{r_{PML}}\right)\left[\frac{x(i)}{x(n_{PML})}\right]^k, \tag{3}$$

where r_{PML} is the maximum absorption rate and the exponent k is used to change the rate of absorption along the absorbing layer. δt is the time step, n_{PML} the number of nodes in the absorbing layer and i is an integer that indicates how far into the layer a given node is ($1 \leq i \leq n_{PML}$). Thus, α ranges from 0 at the first node in the absorbing layer to $ln(10)/B\delta t$ at the boundaries of the domain.

Equations (1) and (2) can be differentiated in time and space and combined to yield a single PML acoustic wave equation,

$$p_{tt} + 2\alpha B p_t + \alpha^2 B^2 p = c^2 \nabla^2 p \,. \quad (4)$$

The Optimized PML

The attenuation coefficient α is defined so that the maximum rate of absorption and the way absorption is inputed can both be changed. Therefore, PML can be tuned to the conditions that are being simulated in order to yield the best results for those conditions.

The reason why PML must be tuned is because inputing absorption creates differences in acoustic impedance in the absorbing layer. That means that as a wave front enters the absorbing layer, it meets different media at each new node it reaches. Therefore, some reflection will happen within the absorbing layer, which is obviously undesirable. Consequently, extremely high absorption rates are undesirable, as represent strong discontinuities in the change in rate of absorption within the layer. As a result, tuning PML to each domain is the only way to make sure that the best compromise between absorption rate and reflection within the absorbing layer is achieved.

Though α originally carries a polynomial function whose exponent is k, there is no limitation as to what kind of function is to be used to change the behaviour of the absorbing layer. Thus, α can be written in a generic form as

$$\alpha(i) = c_{PML} f[x(i)]. \quad (5)$$

This allows for more flexibility and ease in working α to achieve the best possible results in terms of absorption. Throughout this paper, the value of c_{PML} that yields the most absorption for a given setup will be referred to as "best c_{PML}".

The need for impedance matching means that exponent k must be such that the difference in impedance is the smallest possible between each consecutive node in the absorbing layer. However, if k is such that too little absorption is imposed, the overall absorption will not be ideal, which would also happen if c_{PML} were set too small.

This leads to the logical conclusion that the wider the absorbing layer, the smaller c_{PML} should be and the smoother the function defined by k. That way, there will be more room for absorption to happen, which makes it worth it to focus on avoiding large difference in impedance from node to node. Conversely, if the absorbing layer is narrow, a more aggressive setup will yield better results.

Fig. 1 shows a set of possible functions for α. Functions are not, however, limited to the ones shown on the figure.

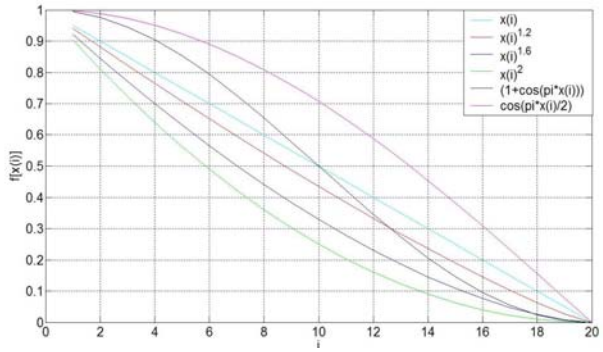


Figure 1. A set of possible functions for α.

Methodology

Several different set ups were tested by changing α and the number of nodes in the absorbing layer. In all simulations, the same Ricker type source was placed at the center of the domain. A 2D, 10th order in space and 2nd order in time finite difference scheme was used. The actual domain was a square with sides ranging from 50 to 600 nodes long, while absorbing layers from 20 to 100 nodes wide were used. In order to ensure numerical stability, the nodes were each 5 meters apart and the time step was set to 0.2 milliseconds. Wave speed was set to 3000 m/s and the medium density to 1000 kg/m³.

Naturally, a measure of energy is necessary to allow comparisons between each simulation; in this work, the sum of the squared amplitudes of all nodes in the 600-node domain was taken on every time step (Fan and Liu, 2000). Mathematically,

$$E = \sum_x \sum_y [U^2(x, y, t)] \cdot \quad (6)$$

Another way of measuring energy can be obtained by summing the results from the first measure for every time step,

$$E = \sum_t \sum_x \sum_y [U^2(x, y, t)] \cdot \quad (7)$$

While the first approach only requires that the domains under comparison be identical in length and width, the second one imposes that they be identical in time span too.

Relationship between c_{PML}, domain size and the width of the absorbing layer

Assessing the relationship between c_{PML}, domain size and the width of the absorbing layer can be very time consuming, since several combinations of the variables in their respective ranges have to be simulated. However, such an analysis can be greatly simplified by introducing a dimensionless number,

$$Z = \frac{number\ of\ nodes\ in\ absorbing\ layer}{number\ of\ nodes\ in\ domain} \,. \quad (8)$$

This approach allows for relevant information to be obtained with relatively fewer simulations. Furthermore, predictions about the ideal value of c_{PML} for a larger domain or absorbing layer can be estimated based on information obtained from a smaller counterpart, whose simulation is much less time-consuming.

Thus, four different domains (50, 75, 100 and 150 nodes) were computed with three different values of Z (1/5, 2/5 and 3/5). Conclusions could then be reached by comparing the twelve different results. The same exponent k (=2) was used for all the simulations.

Results

Fig. 2 shows the ideal value of PML for each combination of domain size (in nodes) and Z. There is a clear tendency towards smaller values of PML both as domains and Z increase, which should be expected from the theory.

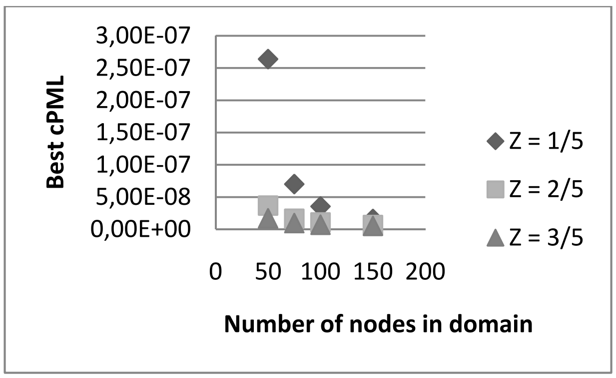


Figure 2. Best c_{PML} for each domain size for k = 2.

Since it is expected that the best c_{PML} diminish as the number of nodes in the domain and/or the absorbing layer increase, the range of values that one needs to consider in searching for the best c_{PML} can be reduced by analyzing smaller domains beforehand. This is an important consequence, because larger domains require considerably more time to simulate.

Fig. 3 shows the computed total energy E for $1 \leq k \leq 10$, where k is an integer; the attenuation coefficient α is given by expression (3). Note that for this case reflection is minimized as $k \rightarrow 4$. Therefore absorption rates are also very sensitive to the polynomial function employed, at least for considerably large absorbing layers.

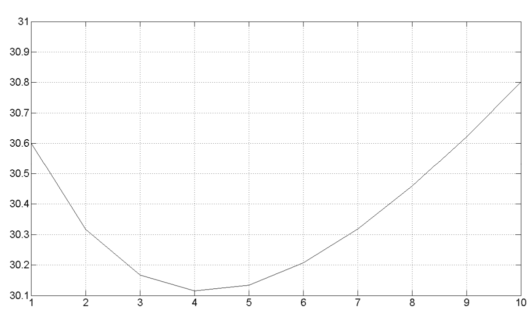
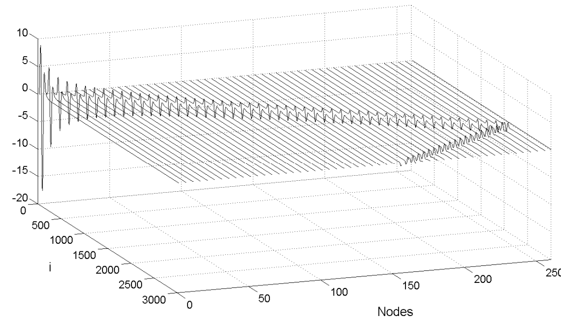
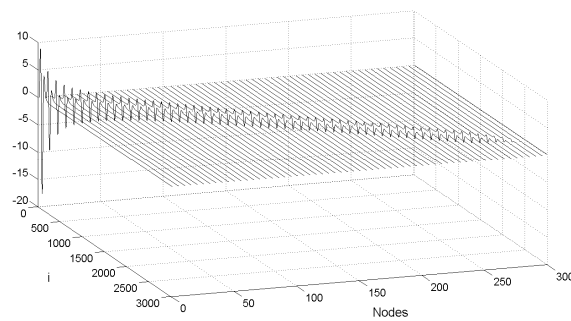


Figure 3. Exponent k versus total energy E.

As the number of PML nodes decreases, numerical results show that the polynomial function becomes less effective for reducing reflection since a transition zone no more exists. If that is the case, c_{PML} becomes the most important PML optimization parameter. Thus Fig. 4 compares time-domain seismograms computed with and without the optimized PML absorbing layer. It is clear from Fig. 4b the absorption rate achieved.



(a)



(b)

Figure 4. Time-domain seismograms computed (a) without and (b) with the optimized PML absorbing layer.

Conclusions

PML was optimized to reduce wave reflections at the borders of the FDTD-2D computational domain. It was found that optimizations increase the effectiveness of the absorbing layer. Results also show that side effects are very sensitive to the number of grid points used in the absorbing layer, with better results found for wider absorbing layers. The relationship between PML's ideal constants and the sizes of the domain and of the absorbing layer was analyzed in order to allow faster optimization of PML; it was shown that the best c_{PML} does, indeed, diminish as the absorbing layer and the domain increase in size.

Acknowledgments

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