Direct Solution of the Acoustic Wave Equation using the Wavelet-Galerkin Method

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Abstract

The use of compactly supported wavelet functions has become increasingly popular in the development of numerical solutions for differential equations. The present work discusses an alternative to the usual finite difference (FDM) approach to the acoustic wave equation modeling by using a space discretization scheme based on the Galerkin Method. The combination of this method with wavelet analysis results in the Wavelet Galerkin Method (WGM) which has been adapted for the direct solution of the wave differential equation in a meshless formulation. This work also introduces Deslauriers-Dubuc wavelets (Interpolets) as interpolating functions. Examples in 1-D were formulated using a central difference (second order) scheme for time differentiation. Encouraging results were obtained when compared with the FDM using the same time steps.

Introduction

Among the numerous techniques available for the solution of the partial differential equation that describes wave propagation, the finite difference approach (Kelly et al, 1976) is by far the most employed one, being used frequently as a standard for the validation of new methods. As a disadvantage, the FDM is known for requiring excessive refining of the model discretization.

The use of wavelet-based numerical schemes has become increasingly popular in the last two decades, especially for problems with local high gradients. Wavelets have several properties that are quite useful for representing solutions of partial differential equations (PDEs), such as orthogonality, compact support and exact representation of polynomials of a certain degree. These characteristics allow the efficient and stable calculation of functions with high gradients or singularities at different levels of resolution (Qian and Weiss, 1992).

A complete basis of wavelets can be generated through dilation and translation of a mother scaling function. Although many applications use only the wavelet filter coefficients of the multiresolution analysis, there are some which explicitly require the values of the basis functions and their derivatives, such as the Wavelet Finite Element Method (WFEM) (Chen et al., 2004).

Compactly supported wavelets have a finite number of derivatives which can be highly oscillatory. This makes the numerical evaluation of integrals of their inner products difficult and unstable. Those integrals are called connection coefficients and they are employed in the calculation of stiffness and mass matrices. Due to some properties of wavelet functions, these coefficients can be obtained by solving an eigenvalue problem using filter coefficients.

Working with dyadically refined grids, Deslauriers and Dubuc (1989) obtained a new family of wavelets with interpolating properties, later called *Interpolets*. Their filter coefficients are obtained from the autocorrelation of the Daubechies' coefficients (Daubechies, 1988). In consequence, interpolets are symmetric, which is especially interesting in numerical analysis. The use of interpolets instead of Daubechies' wavelets considerably improves the method's accuracy.

In this work, the Wavelet-Galerkin Method has been adapted for the direct solution of differential equations in a meshless formulation. This approach enables the use of a multiresolution analysis. Accuracy can be improved by increasing either the level of resolution or the order of the wavelet used.

As a preliminary study, the formulation of an interpolet-based Galerkin scheme was demonstrated for a one-dimensional wave propagation problem. Some examples were formulated and results compared with the standard Finite Differences Method.

Interpolets

Multi-resolution analysis using orthogonal, compactly supported wavelets has become increasingly popular in numerical simulation. Wavelets are localized in space, which allows the analysis of local variations of the problem at various levels of resolution.

In the following expression, known as the two-scale relation, a_k are the filter coefficients of the wavelet scale function.

$$\varphi(x) = \sum_{k=1-N}^{N-1} a_k \varphi(2x-k) = \sum_{k=1-N}^{N-1} a_k \varphi_k(2x)$$

The basic characteristics of interpolating wavelets require that the mother scaling function satisfies the following condition (Shi et al, 1999):

$$\varphi(k) = \delta_{0,k} = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \quad k \in \mathbb{Z}$$

The Deslauriers-Dubuc (1989) interpolating function of order *N* is given by an autocorrelation of the Daubechies' scaling filter coefficients (h_m) of the same order (i.e. $N/2$ vanishing moments). Its support is given by [1-N,N-1], it has even symmetry and is capable of representing polynomials of order up to N-1.

$$a_k = \sum_{m=0}^{N-1} h_m h_{m-k}$$

Interpolets satisfy the same requirements as other wavelets, specially the two-scale relation, which is fundamental for their use as interpolating functions in numerical methods. Figure 1 shows the interpolet IN6 (autocorrelation of DB6). Its symmetry and interpolating properties are evident. There is only one integer abscissa which evaluates to a non-zero value.

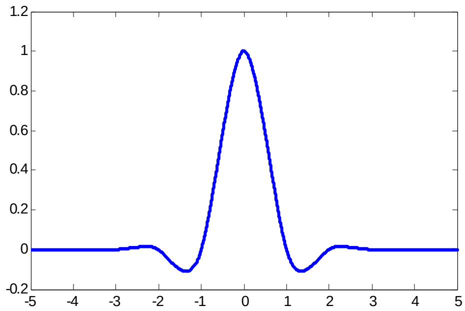

Figure 1: Interpolet IN6 scaling function with its full support.

The numerical solution of differential equations is one of the possible applications of the wavelet theory. The Wavelet-Galerkin Method (WGM) results from the use of wavelets as interpolating functions in a traditional Galerkin scheme (Du & Bancroft, 2004). In the following sections, the WGM will be applied to solve the typical DE for acoustic wave propagation.

Wave Propagation using the WGM

The partial differential equation (PDE) which rules the wave propagation in 1-D is:

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$

where, u is the horizontal displacement and α is the medium velocity. Assuming that the displacement u is approximated by a series of interpolating scale functions, the following may be written:

$$u(\xi) = \sum_{k=2-N}^{N-1} d_k \varphi(\xi - k)$$

The Galerkin method consists in substituting the expression above in the differential equation and forcing the approximation error to be orthogonal to a test result which is formulated using the same interpolating functions.

$$\alpha^2 \left\{ \int_0^1 \mathbf{\Phi}^T \mathbf{\Phi}'' d\xi \right\} \mathbf{d} = \left\{ \int_0^1 \mathbf{\Phi}^T \mathbf{\Phi} d\xi \right\} \ddot{\mathbf{d}}$$

Using this approach, the PDE can be rewritten at a specific time t as a system of linear equations, which in matrix form is:

$$\mathbf{m}\,\ddot{\mathbf{d}}_t + \mathbf{k}\mathbf{d}_t = \mathbf{0}$$

In this expression, **m** represents the mass matrix and **k** is the stiffness matrix of the model, which in normalized coordinates (ξ) within the interval [0,1] are given by:

$$k_{i,j} = \alpha \int_0^1 \varphi_i'(\xi)\varphi_j'(\xi) d\xi = \alpha\, \Lambda_{i,j}^{1,1}$$

$$m_{i,j} = \int_0^1 \varphi_i(\xi)\varphi_j(\xi) d\xi = \Lambda_{i,j}^{0,0}$$

The so-called connection coefficients Λ appear in the expressions above. Wavelet dilation and translation properties allow the calculation of connection coefficients to be summarized by the solution of an eigenvalue problem based only on filter coefficients (Zhou & Zhang, 1998).

$$\left(\mathbf{P} - \frac{1}{2^{d_1+d_2-1}}\mathbf{I}\right)\mathbf{\Lambda}^{\mathbf{d_1},\mathbf{d_2}} = \mathbf{0}$$

$$\mathbf{P} = \left[a_{r-2i}a_{s-2j} + a_{r-2i+1}a_{s-2j+1}\right]_{i,j,r,s=(2-N)\ldots(N-1)}$$

Since the expression above leads to an infinite number of solutions, there is the need for a normalization rule that provides a unique eigenvector. This unique solution comes with the inclusion of the so-called moment equation, derived from the wavelet property of exact polynomial representation (Burgos et al., 2009).

As in the FDM, it becomes necessary to solve the system of equations at discrete time intervals. There are several effective direct integration methods, among which the most intuitive one is the Central Difference Method:

$$\ddot{\mathbf{d}}_t \cong \frac{\mathbf{d}_{t+\Delta t} - 2\mathbf{d}_t + \mathbf{d}_{t-\Delta t}}{(\Delta t)^2}$$

Substituting the expression of the acceleration obtained by the Central Difference Method and solving for the next time step $\mathbf{d}_{t+\Delta t}$:

$$\mathbf{m}\mathbf{d}_{t+1} = \mathbf{m}(2\mathbf{d}_t - \mathbf{d}_{t-1}) - (\Delta t)^2 \mathbf{k}\mathbf{d}_t$$

Stability of the Central Difference Method is conditioned to the choice of the time step, whose upper bound is obtained from a generalized eigenvalue problem.

$$(\mathbf{k} - \omega^2\mathbf{m})\mathbf{d} = \mathbf{0} \rightarrow (\mathbf{X} - \omega^2\mathbf{I})\mathbf{d} = \mathbf{0}$$

$$\Delta t_{\max} = \frac{\sqrt{2}}{\omega_{\max}}$$

Matrix **m** might not be invertible. In this case, boundary conditions shall be imposed by using Lagrange multipliers. This procedure leads to a square matrix which can be useful for some system solvers.

$$\begin{bmatrix} \mathbf{m} & \mathbf{g}^T \\ \mathbf{g} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{d}_{t+1} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{m}(2\mathbf{d}_t - \mathbf{d}_{t-1}) - (\Delta t)^2 \mathbf{k}\mathbf{d}_t \\ \mathbf{0} \end{Bmatrix}$$

In the expression above, the matrix **g** is associated with boundary conditions and λ is a vector of Lagrange multipliers which is not used in the solution. The main difference in relation to the FDM is that the unknowns in vector **d** are the interpolating coefficients of the basis functions instead of nodal displacements. In fact, there is no need to establish nodal coordinates.

When dealing with one-dimensional problems, most wavelets (including Daubechies and Interpolets) present a mass matrix whose rank is one unit less than its size. This means that only one boundary condition needs to be imposed for the system to have a solution.

Examples

To validate the formulation, a 1-D example was implemented, consisting in applying a ricker source at the midpoint of a pinned, unit length rod. The propagation was modeled by the FDM using 265 points and Δt=0.3*ms*, with fourth and second order discretization in space and time, respectively. This time step was obtained using the upper bound described in the previous section. For this example, the lowest central frequency of the source that produces a numerical dispersion is ω=80Hz. The same source central frequency, special discretization and time step were used in the WGM example, with no visible dispersion in the results. Figure 2 shows the response at time t=0.45*s* for both methods using a source central frequency of ω=40Hz. There is no visible numerical dispersion in either case. Figure 3 shows the response using a source central frequency of ω=80Hz, which produces numerical dispersion in the FDM model.

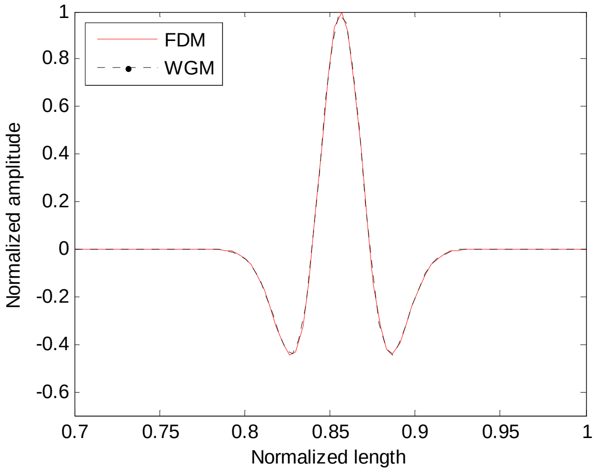

Figure 2: FDM and WGM results for ω=40Hz

As a second example, the rod is made by two different materials and the dispersion in the case of the FDM is even greater, as shown in figure 4. As expected, the change in velocity introduces additional errors in the FDM model. These errors are not present in the WGM model.

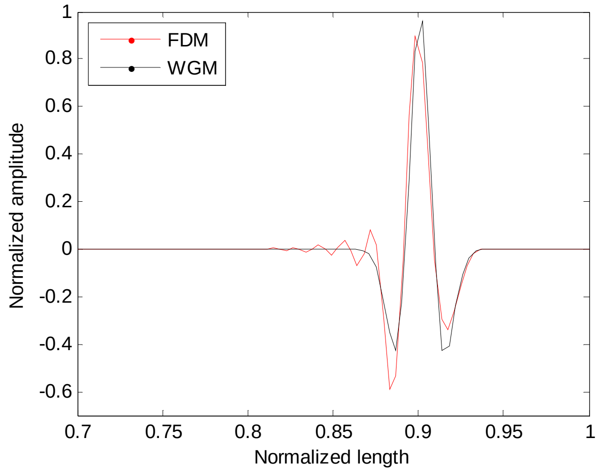

Figure 3: FDM and WGM results for ω=80Hz

Conclusions

This work presented the formulation and validation of the Wavelet-Galerkin Method (WGM) using Deslauriers-Dubuc Interpolets. These preliminary results are promising, but the simplicity of the studied models has to be taken into account. The main improvement in the presented formulation was the recognition of a different dispersion pattern when comparing FDM and WGM results using the same space and time grid. Both methods used second order time discretization and the FDM used fourth order space discretization, which shows that comparisons were made against a robust numerical scheme.

All matrices involved can be stored and operated in a sparse form, since most of their components are null, thus saving computer resources. Due to the compact support of wavelets, the sparseness of matrices increases along with the level of resolution.

In future works, models with greater complexity will be analyzed and different families of wavelets will be explored. The extension of the method to irregular geometries in two-dimensional problems is still a challenge, but one potential advantage is the possibility of implementing absorbing boundary conditions analytically with the use of Lagrange Multipliers.

Acknowledgments

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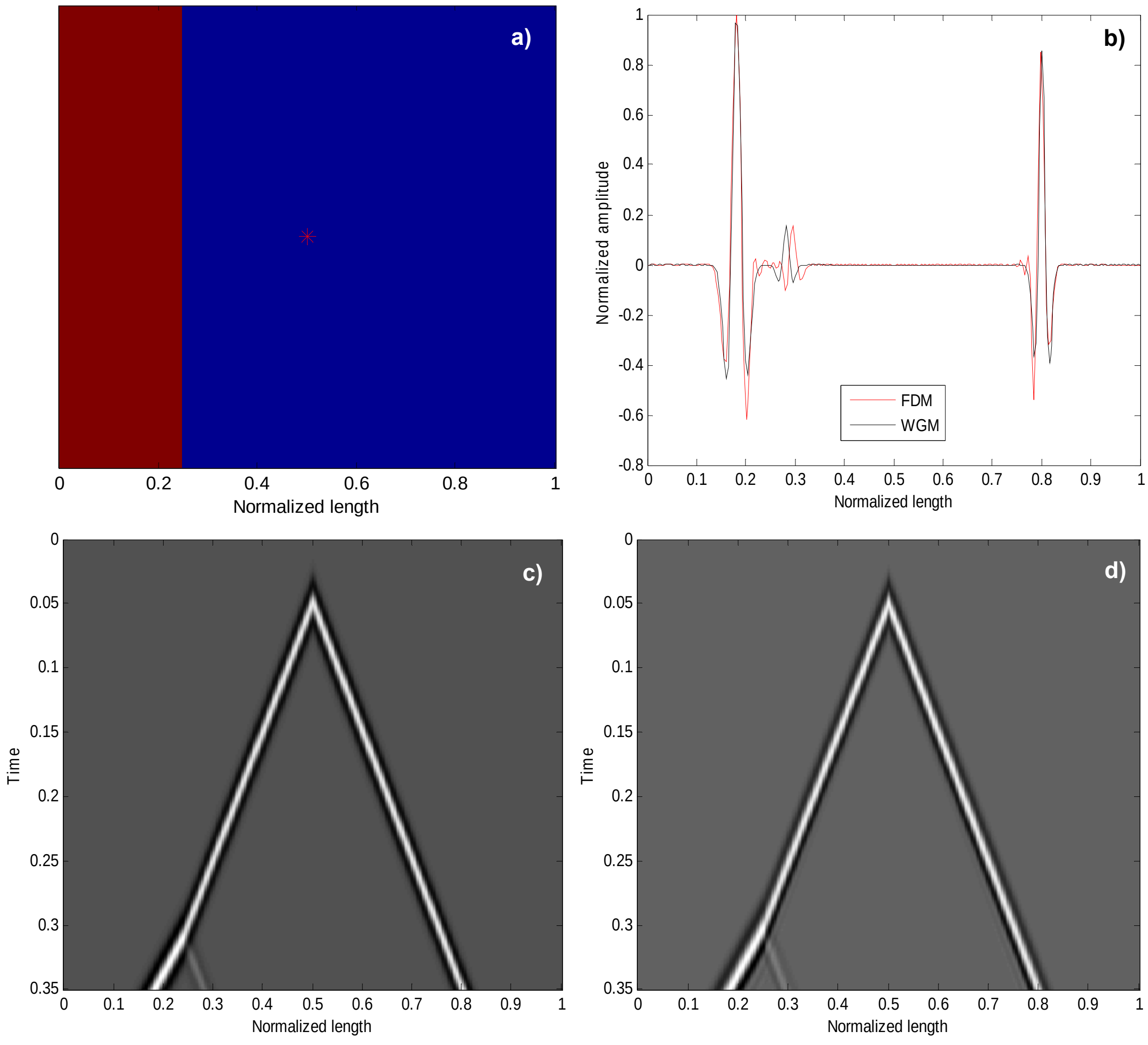

Figure 4: (a) Velocity profile (red region is a faster medium) and source position; (b) Snapshot for amplitude comparison at time t=0.35*s* for example 2; (c) Seismogram obtained using WGM; (d) Seismogram obtained using FDM with noticeable numerical dispersion.