Electromagnetic modelling using unstructured grid Finite Difference Method and Radial Basis Functions

Diego C. Miranda, Allen Q. Howard Jr., Universidade Federal do Pará, Brazil

Abstract

We assess the Finite Difference frequency domain method to modelling the electromagnetic response of energized geological media. We start from the 2.5D electromagnetic formulation using secondary scalar and vector electromagnetic potentials. The use of secondary potentials removes singular problems with primary fields. To model rapidly changing fields we also tested non-uniform unstructured grids and show that the use of radial basis functions can handle, with good accuracy, the numerical derivatives that arise from our procedure.

Introduction

When dealing with electromagnetic geophysical problems, the expressions of fields and potentials come from the solution of differential equations. In the most complex cases, it is necessary to use numerical methods to generate an approximate solution, since for most geometries it is impossible to obtain an analytical expression. Some of the most common numerical methods are for example, the finite difference method (Mackie et al. 1993), the finite element method (Schenkel, 1991) and the integral equation method (Hohmann,1975).

The Finite Difference method (FD) is a numerical procedure used to solve differential equations in which the derivatives are approximated by finite differences. One of its main advantages is its easy implementation, making it suitable for a wide variety of problems. Finite Difference has generally been carried out using regularly structured grids (Franke et al. 2004), which facilitates the calculation of numerical derivatives but causes it to be less flexible in the task of delineating complex structures.

Methods, such as finite element and integral equation, are very popular in areas like engineering and physics because of their flexibility and versatility, and this is partly due to their ability of using adaptive meshes, however, they have a serious limitation in efficiency for large number of variables (Fernandez & Kulas, 2004). Adaptive meshes are very appropriate in cases where the solution sought varies widely in the problem domain or with boundaries non-conforming to the grid, which is the case of most of the problems in geophysics.

In this paper, we present the 2.5D electromagnetic formulation, in terms of the secondary electrical scalar and magnetic vector potentials to be computed by the Finite Difference method. We have used unstructured grid, created by a finite element mesh generator (Persson, 2005), and to derive first and second order partial derivatives at each node, we used the expression of a radial basis function, calculated in terms of the function values at a number of neighboring mesh nodes. Thus, no spatial interpolation has to be performed between unstructured and uniform meshes.

2.5D electromagnetic formulation

We start from coupled Maxwell's equations in the frequency domain, as showed below

$$\nabla\times\mathbf{H}(x) = \big(\sigma(x)-i\omega\varepsilon\big)\mathbf{E}(x)+\mathbf{J}_s(x) \quad (1)$$
$$\nabla\times\mathbf{E}(x) = +i\omega\mu\mathbf{H}(x) \quad (2)$$

where σ is the electrical conductivity, ω is the angular frequency, $\mathbf{J}_s$ is the current density and μ is the magnetic permeability. Then, we define the inhomogeneous wavenumber $k(x)$ as

$$k(x)=\sqrt{\omega^2\mu_0\big(\varepsilon+i\sigma(x)/\omega\big)}, \qquad \mathrm{Im}(k(x))>0. \quad (3)$$

Here we uncouple $\mathbf{E}(x)$ and $\mathbf{H}(x)$ fields by introduction of the magnetic vector potential $\mathbf{A}(x)$, which defines the magnetic vector induction:

$$\mathbf{B}(x)=\nabla\times\mathbf{A}(x) \quad (4)$$

Using (4) in (1) and (2), yields

$$\mathbf{E}(x)=i\omega\mathbf{A}(x)-\nabla\phi(x) \quad (5)$$

And from conservation of charge $\nabla\cdot\mathbf{J}(x)-i\omega\rho(x)=0$, we can obtain, after some manipulation, the following equations

$$\Big(\nabla^2+k^2(\mathbf{x})-K_z^2\Big)\mathbf{A}_s(\mathbf{x})+2\mu_0\nabla\alpha(\mathbf{x})V_s(\mathbf{x}) = -\big(k^2(\mathbf{x})-k_1^2\big)\mathbf{A}_0(\mathbf{x})-2\mu_0\nabla\alpha(\mathbf{x})V_0(\mathbf{x}) \quad (6)$$

and

$$\left(\nabla^2+k^2(\mathbf{x})-K_z^2-\frac{\nabla^2\alpha(\mathbf{x})}{\alpha(\mathbf{x})}\right)V_s(\mathbf{x})-2i\omega\nabla\alpha(\mathbf{x})\cdot\mathbf{A}_s(\mathbf{x}) = 2i\omega\nabla\alpha(\mathbf{x})\cdot\mathbf{A}_0(\mathbf{x})+\left(k^2(\mathbf{x})-k_1^2-\frac{\nabla^2\alpha(\mathbf{x})}{\alpha(\mathbf{x})}\right)V_0(\mathbf{x}). \quad (7)$$

Equations (6) and (7) form the coupled linear system related to the secondary potentials $\mathbf{A}(x)$ and $V(x)$, where the primary potentials were subtracted to avoid source singularity problems. The scalar potential $V(x)$ is related to the scalar potential $\phi(x)$ by the expression:

$$\phi(\mathbf{x}) = \frac{V(\mathbf{x})}{\alpha(\mathbf{x})}, \tag{8}$$

where

$$\alpha(\mathbf{x}) = \sqrt{\sigma(\mathbf{x})}. \tag{9}$$

This transformation of the scalar potential renders the operators symmetric in equation (7) as required when using for example, the conjugate gradient method as solver.

The resolution of the linear system by Finite Difference method requires a complete discretization of our domain, yielding a very large sparse matrix to be solved. The unstructured grid is refined in the regions where the fields vary more rapidly.

Handling numerical derivatives

Once the unstructured grid is established, the idea is to fit a surface of a given order to each grid node, with the aid of some neighboring nodes. Thus, a surface can be defined by:

$$f(x) = \sum_{j=1}^{n} \lambda_j \phi(\|x - x_j\|), \tag{10}$$

where $\phi(r)$, $r \geq 0$ is some radial function and n is the number of neighboring nodes plus the central node. The coefficients λ_j are determined from the condition $f(x_j) = s_j$, $j = 1, \ldots, n$, where s_j is the value of the function at each of the n nodes, which leads to the follow symmetric linear system:

$$\begin{bmatrix} & A & \end{bmatrix} \cdot \begin{bmatrix} \lambda \end{bmatrix} = \begin{bmatrix} f \end{bmatrix}, \tag{11}$$

where the entries of A are given by $a_{j,k} = \phi(\|x_j - x_k\|)$. The procedure described above is knows as Basic Radial Basis function (RBF) Method, and a good account of its formulation can be found in Wright (2003). Some common examples of the $\phi(r)$ that lead to a uniquely solvable method are given in table 1.

Type of basis function	$\phi(\mathbf{r})(\mathbf{r} \geq \mathbf{0})$
Gaussian (GA)	$e^{-\varepsilon^2 r^2}$
Inverse quadratic	$\frac{1}{1+(\varepsilon r)^2}$
Inverse multiquadratic	$\frac{1}{\sqrt{1+(\varepsilon r)^2}}$
Multiquadratic	$\sqrt{1+(\varepsilon r)^2}$

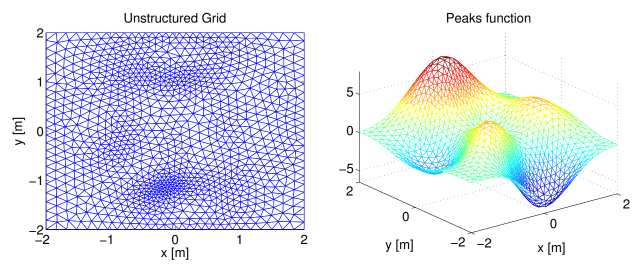
Table 1: Examples of radial basis functions.

The parameter ε is some fixed non-zero value that controls the shape of functions. In our problem, we use the Gaussian function as $\phi(r)$, and for this expression, first and second order derivatives are give by:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \sum_{j=1}^{n} \lambda_j \left[-2\varepsilon^2 (x - x_j) e^{-\varepsilon^2 r^2}\right] \\
\frac{\partial f}{\partial y} &= \sum_{j=1}^{n} \lambda_j \left[-2\varepsilon^2 (y - y_j) e^{-\varepsilon^2 r^2}\right] \\
\frac{\partial^2 f}{\partial x^2} &= \sum_{j=1}^{n} \lambda_j \left[-4\varepsilon^2 [1 - \varepsilon^2 (x - x_j)^2] e^{-\varepsilon^2 r^2}\right] \\
\frac{\partial^2 f}{\partial y^2} &= \sum_{j=1}^{n} \lambda_j \left[-4\varepsilon^2 [1 - \varepsilon^2 (y - y_j)^2] e^{-\varepsilon^2 r^2}\right]
\end{aligned} \tag{12}$$

where $r = \sqrt{(x - x_j)^2 + (y - y_j)^2}$.

To evaluate the performance of RBF method, we use the MATLAB® *peaks function* and its first and second order analytical derivatives to compare with the numerical results we obtained. We computed numerical first and second order derivatives of peaks function using the RBF method and also using the procedure where we interpolate the peaks function from the unstructured to uniform grid and then calculate the derivatives. Finally, we compare each result with its corresponding analytical through the absolute difference of both



.

Figure 1: *a*) Adaptive unstructured grid refined in the regions of greatest variation of f. *b*) Peaks function

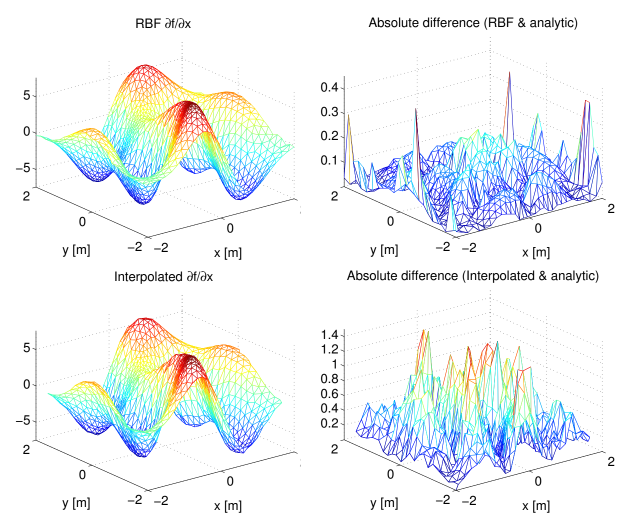


Figure 2: First order derivatives comparison.

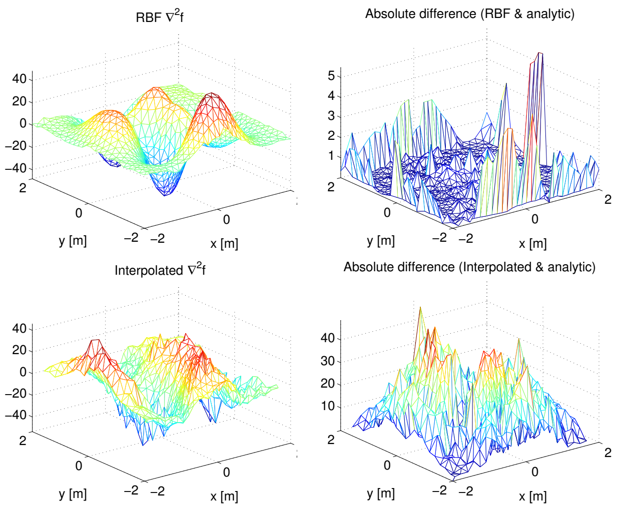


Figure 3: Second order derivatives comparison.

Figure 2 shows that, although the interpolation method has generated an reasonable result, it is less accurate than the result obtained by the RBF method. And in Figure 3, we noticed that for the calculation of the second derivative, the interpolation method is totally impaired, whereas the RBF method still generates good results.

A geophysical example

In the following example, we model the flow of electric current between two media with different electrical conductivities. For that, we refine the unstructured grid in the interface between the two media, where the electrical fields have a rapid variation, which is the response of the electrical current flowing between two source electrodes placed near and parallel to the interface between the media. Because the lines of current flow are always parallel to the equipotential surfaces, the arrangement of these surfaces for this type of source indicates that there will be current flowing between the two media, as shown in Figure 4.

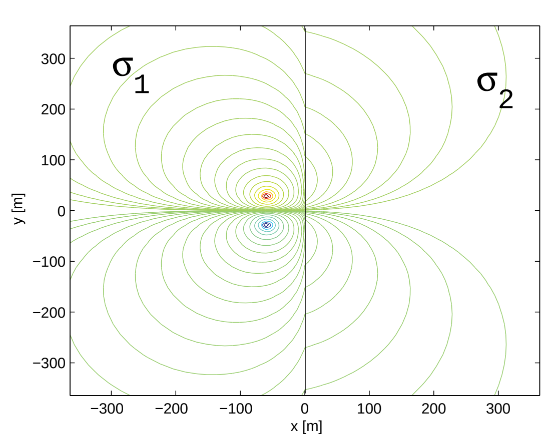


Figure 4: Representation of equipotentials surfaces.

For the *dc* current case, equations (6) e (7) reduce to:

$$\left(\nabla^2 - K_z^2 - \frac{\nabla^2\alpha(\mathbf{x})}{\alpha(\mathbf{x})}\right) V_s(\mathbf{x}) = -\left(\frac{\nabla^2\alpha(\mathbf{x})}{\alpha(\mathbf{x})}\right) V_0(\mathbf{x}). \qquad (13)$$

Solving the linear system of equations generated by the discretization of equation (13) in the domain and performing the inverse transformation of equation(8) in the resulting V_s, give us the secondary scalar potencial $\phi_s(x,y)$, as follow:

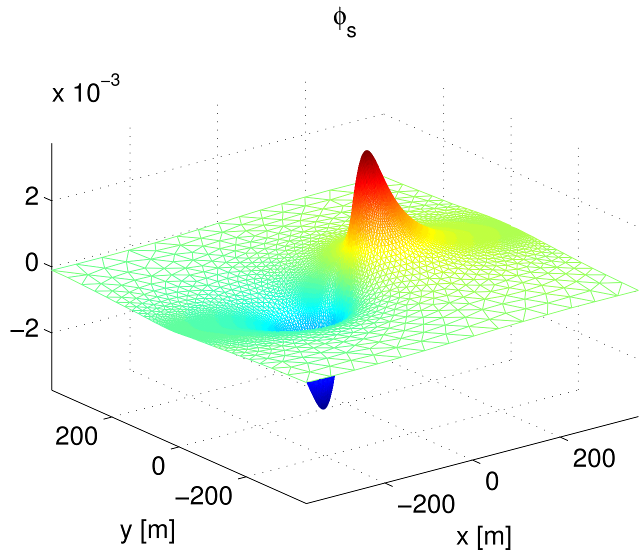


Figure 5: Numerical secondary electrical scalar potential.

To validate this result, we compare it with the analytical solution of this problem, which can be found for example in Wait (1982).

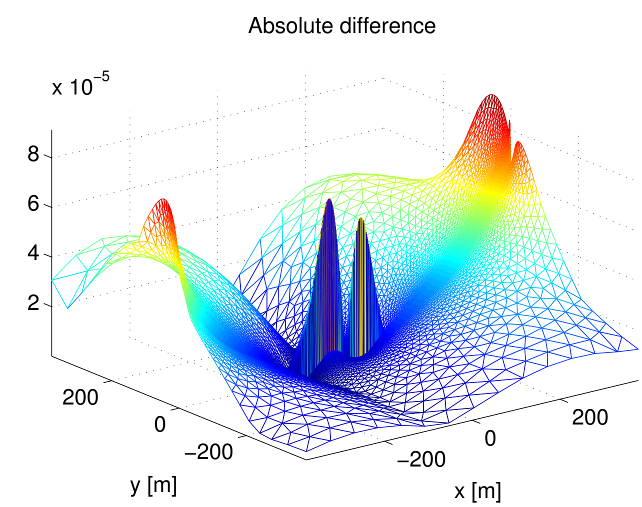


Figure 6: Absolute difference between numerical and analytical solution for the scalar wave problem.

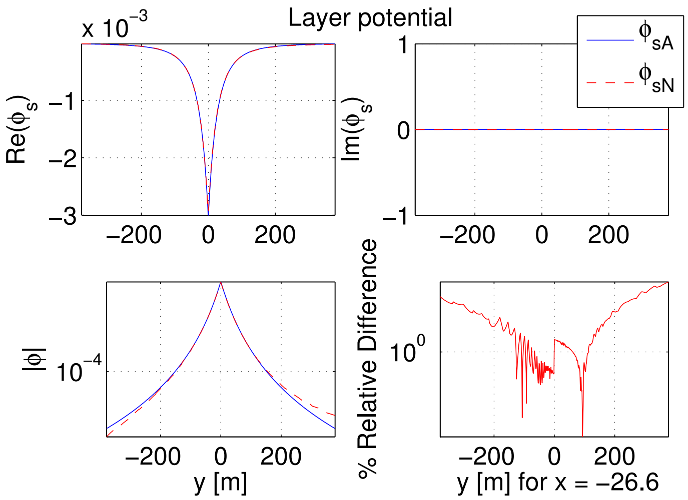


Figure 7: Absolute difference between numerical and analytical solution for the scalar wave problem.

Figure 6 shows the absolute difference between the numerical and analytical solution of the presented problem. Figure 7 compares results taking the potential values on a line perpendicular to the media interface. We notice that the numerical solution becomes worse when approaching the edges. This occurs because the finite difference method is less accurate in this region, where the calculation of the derivatives is affected due to the lower number of neighboring points that we can choose. Results show a good approximation obtained by the procedure.

Summary, Comments and Conclusions

We present the development for the modelling of the electromagnetic potentials by unstructured grid Finite Difference method. We have shown that is possible to obtain a good approximation of the derivatives that arise in the problem through the use of radial basis functions, with no need of spatial transformations between unstructured and structured grids. The derivatives evaluation with the RBF method proved to be well suited even in functions with certain discontinuities, although this can be improved by the use of filters to smooth discontinuous regions. Results show considerable advantages over the standard Finite Difference method and indicate that the methodology can be very effective in more complex problems, which are our future extensions.

References

Fernandez, F. A. and Kulas, L., 2004. A simple finite difference approach using unstructured meshes from fem mesh generators. In: 15th International Conference on Microwaves, Radar and Wireless Communications, MIKON-2004,585-588).

Franke, A., Börner, R. and Spitzer, K., 2004. 2D Finite Element modelling of plane-wave diffusive time-harmonic electromagnetic fields using adaptive unstructured grids, IAGA WG 1.2 on Electromagnetic Induction in the Earth, Expanded Abstract, S.2-O.1.

Wright, G. B., 2003. Radial Basis Function Interporlation: Numerical and Analytical Developments. Ph.D. Thesis, University of Colorado.

Hohmann, G. W., 1975. Three-dimensional induced polarization and electromagnetic modelling. Geophysics, 40, 309-324.

Mackie, R. L., Madden T. R. and Wannamaker P. E., 1993. Three-dimensional mangnetotelluric modelling using difference equations-Theory and comparisons to integral equation solutions. Geophysics, 58, 215-226.

Persson, P., 2005. Mesh Generation for Implicit Geometries. Ph.D. Thesis, Masshachusetts Institute of Technology.

Schenkel, C. J., 1991, The Electrical Resistivity Method in Cased Boreholes. Ph.D. Thesis, University of California.

Wait, J. R. 1982. Geo-electromagnetism. Academic Press, New York.

Acknowledgments

We would like to thank CAPES, ANP and PETROBRAS for their support in this research.