Finite-Differences with Adaptive Spatial Operator for the 1D Wave Equation: Dispersion and Stability Analysis

Alexandre William Camargo & Lúcio Tunes Santos, University of Campinas (UNICAMP), Brazil

Abstract

Most of Finite Difference (FD) methods used in seismic modeling are based on fixed length spatial operators. These operators are chosen observing computacional cost, stability and dispersion criteria. In this work we analyse a FD scheme with an adaptive spatial operator which reduces the computational cost but not the accuracy. The idea is to use long operators in low velocity regions and short operators in high velocity ones. The analysis is made in the unidimensional case, but the results can be extended for 2D and 3D models.

Introduction

Seismic modeling simulates the wave propagation in subsurface. One of the fundamental basis of seismic modeling is the acoustic wave equation which requires, with rare exceptions, efficient numerical methods to be solved. Due to their simple implementation, algorithms based on Finite Differences (FD) are preferred to solve the acoustic wave equation (Liu and Sen, 2009, 2011; Dablain, 1986; Kelly et al., 1976). In addition, if a FD method satisfies all the criteria of stability and dispersion, the numerical solution is of excelent quality.

If we are not concerned with computacional costs, we can use a spatial grid based on the the minimum velocity. As the source usually is fixed in the modeling, its main frequency is also fixed and then the wavelength in the region with low velocity is smaller than in the region with high velocity. Therefore, the accuracy is greater in the high velocity regions. There are many variants of FD method to increase efficiency without decreasing accuracy, or to increase accuracy without decreasing efficiency or to increase both efficiency and accuracy see, e.g., (Virieux, 1984, 1986; Finkelstein and Kastner, 2007; Bartolo et al., 2012).

For fixed spatial and time steps, when we use the same length for the spatial operator of the FD scheme, we reach greater accuracy in higher velocity regions than in lower ones. Therefore, it would be more efficient if we could choose the length of the spatial operator according to the velocity. In this work we analyse one approach introduced by Liu and Sen (2011), based on a FD scheme with an adaptive spatial operator. The length of the spatial operator is chosen based on the analysis of the stability and dispersion in each velocity region, in such a way that the length decreases with increasing velocity. Numerical examples illustrate the approach.

Finite Difference Methods

Let $x \in \mathbb{R}^n$ $(n = 1,2,3)$ the space variable , $t \in \mathbb{R}$ the temporal variable and c the function which describes the velocity of propagation in the acoustic model. The homogeneous (without source term) acoustic wave equation is given by

$$\frac{1}{c(x)^2} U_{tt}(x,t) - \Delta U(x,t) = 0, \tag{1}$$

where $U(x,t)$ is the scalar wavefield, and Δ denotes the Laplacian operator. We will analyse the case $n = 1$, but the results can be easily extended for $n = 2$ and $n = 3$. Applying the a second-order FD operator in time and a $(2M)$th-order FD operator in space, we get

$$\frac{\partial^2}{\partial t^2} U(x_j,t_k) \approx \frac{1}{\Delta t^2}\left[-2u_j^k + (u_j^{k-1} + u_j^{k+1})\right], \tag{2}$$

$$\frac{\partial^2}{\partial x^2} U(x_j,t_k) \approx \frac{1}{\Delta x^2}\left[a_{0,j}u_j^k + \sum_{m=1}^{M} a_{m,j}(u_{j-m}^k + u_{j+m}^k)\right], \tag{3}$$

where $u_j^k = U(x_j,t_k)$, $x_j = x_0 + j\Delta x$, $t_k = k\Delta t$, with $j = 0,1,\ldots,J$ and $k = 0,\ldots,K$. The coefficients $a_{m,j}$ are given by (Liu and Sen, 2009; Finkelstein and Kastner, 2007)

$$a_{m,j} = \frac{(-1)^{m+1}}{m^2} \prod_{n=1,n\neq m}^{M} \left|\frac{n^2 - r_j^2}{n^2 - m^2}\right|, \quad m = 1,2,\ldots,M, \tag{4}$$

and

$$a_{0,j} = -2\sum_{m=1}^{M} a_{m,j}, \tag{5}$$

where $r_j = c(x_j)\Delta t/\Delta x$ are the Courant numbers. Substituting equations (2) and (3) into equation (1), we obtain the following recursion formula,

$$u_j^{k+1} = 2u_j^k - u_j^{k-1} + r_j^2\left[a_{0,j}u_j^k + \sum_{m=1}^{M} a_{m,j}(u_{j-m}^k + u_{j+m}^k)\right]. \tag{6}$$

Dispersion and Stability Analysis

The dispersion occurs when the phase velocity v_j and velocity c_j are different. The difference is measured by the ratio between them, given by

$$\psi_j = \frac{v_j}{c_j} = \frac{\varphi_j/\xi}{c_j}, \tag{7}$$

where ξ is the wavenumber and φ_j is the dispersion angular frequency determined by plane wave theory applied to the recursion formula (6). Taking

$$u_j^k = e^{i(\xi x_j - \varphi_j t_k)}, \tag{8}$$

and substituting it into equation (6), we find

$$\varphi_j = \frac{2}{\Delta t} \arcsin \sqrt{r_j^2 \sum_{m=1}^{M} a_{m,j} \sin^2\left(\frac{m\gamma}{2}\right)}, \tag{9}$$

where $\gamma = \xi \Delta x$. Therefore,

$$\psi_j = \frac{2}{r_j \gamma} \arcsin \sqrt{r_j^2 \sum_{m=1}^{M} a_{m,j} \sin^2\left(\frac{m\gamma}{2}\right)}. \tag{10}$$

The stability of the method can be obtained by the eigenvalue analysis of the transition matrix G_j, defined from the relation

$$\begin{pmatrix} u_j^k \\ u_j^{k-1} \end{pmatrix} = G_j \begin{pmatrix} u_j^{k-1} \\ u_j^{k-2} \end{pmatrix}, \tag{11}$$

which can be obtained by the usual Von Neumman analysis of equation (6). The result is

$$G_j = \begin{pmatrix} g_j & -1 \\ 1 & 0 \end{pmatrix}, \tag{12}$$

where

$$g_j = 2 + 2r_j^2 \sum_{m=1}^{M} a_{m,j}[\cos(m\gamma) - 1]. \tag{13}$$

The eigenvalues of G_j are given by

$$\alpha_{\pm} = \frac{g_j \pm \sqrt{g_j^2 - 4}}{2}. \tag{14}$$

Therefore, if $|g| < 2$ we have $|\alpha_{\pm}| < 1$, and then the recursion is stable. Assuming that, in general, the error increases with the wavenumber, we can consider the Nyquist wavenumber

$$\xi_{Nyq} = \frac{\pi}{\Delta x} \tag{15}$$

as the maximum value for ξ. Hence, for that value

$$g_j = 2 - 4r_j^2 \sum_{m=1}^{M_1} a_{2m-1,j}, \tag{16}$$

where M_1 is the integer part of M. Concluding, the one-dimensional stability condition is

$$\left| 2 - 4r_j^2 \sum_{m=1}^{M_1} a_{2m-1,j} \right| < 2. \tag{17}$$

Since from equation (4) we have $a_{2m-1,j} > 0$, the method is stable for

$$r_j < \left(\sum_{m=1}^{M_1} a_{2m-1,j} \right)^{-1/2}. \tag{18}$$

In Liu and Sen (2009) they show that for $r \leq 1$ the above inequality is satisfied.

Figures 1 and 2 depicts the dispersion curves, i.e., the variation of ψ with γ, in the cases $M = 2$ and $M = 16$ for some values of r in the interval $(0,1)$. One can observe that the region for γ that makes $\psi \approx 1$ extends with the increasing of M.

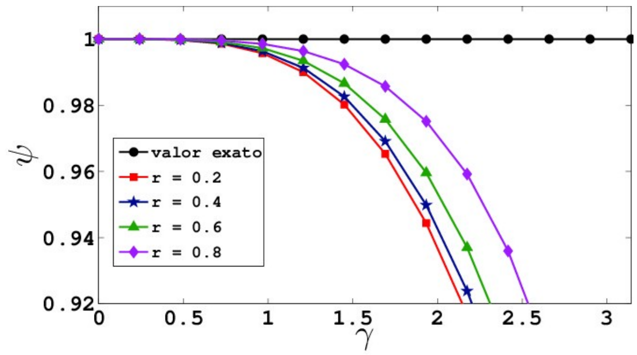

Figure 1: Dispersion curves for $M = 2$.

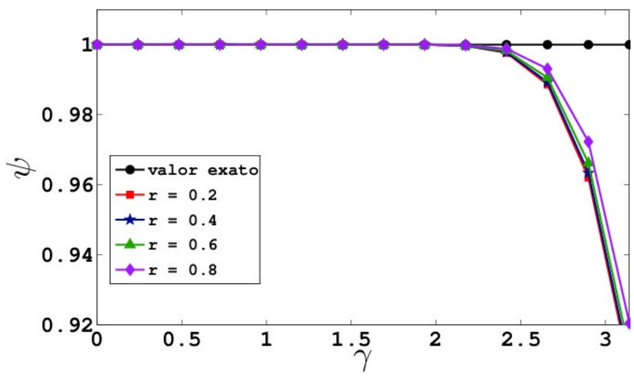

Figure 2: Dispersion curves for $M = 16$.

The Choice for Δx, Δt and M

Before presenting the method that chooses the length operator, we will explain why Δx can be fixed without loss of accuracy or stability. Based on Figures 1 and 2, for given $\varepsilon > 0$ and $M \geq M_{min} \geq 1$, there exist γ_{max} and r_{max} such that,

$$|\psi - 1| \leq \varepsilon \quad \text{if} \quad \gamma \leq \gamma_{max} \quad \text{and} \quad r \leq r_{max}. \tag{19}$$

Denoting by λ the wavenumber and f the frequency, we can write

$$\gamma = \xi \Delta x = \frac{2\pi \Delta x}{\lambda} = \frac{2\pi f \Delta x}{c}. \tag{20}$$

Assuming that the maximum frequency is f_{max} and the minimum velocity is c_{min}, we have that

$$\gamma \leq \gamma_{max} \quad \text{if} \quad \Delta x \leq \frac{\gamma_{max} c_{min}}{2\pi f_{max}}. \tag{21}$$

Once Δx is chosen, Δt can be given by

$$\Delta t \leq \frac{\Delta x \, r_{max}}{c_{max}}, \tag{22}$$

where c_{max} is the maximum velocity. Remember that for the FD scheme to be stable is necessary that $r_{max} < 1$.

For example, let us consider the case of $f_{max} = 30$ Hz, $c_{min} = 1.5$ km/s, $c_{max} = 4$ km/s, and the tolerance for the dispersion $\varepsilon = 0.05$. For $M_{min} = 2$, from Figure 1 we can choose $\gamma_{max} = 2$ and $r_{max} = 0.5$. Therefore, we can take $\Delta x \approx 15$ m and $\Delta t \approx 2$ ms.

Now, let us explain how to choose the length operator M_j according to the velocity c_j. For a fixed M the error in the

FD scheme can be measured by the difference between FD and exact propagation times,

$$\mu(M,c_j) = \frac{\Delta x}{v_j} - \frac{\Delta x}{c_j} = \frac{\Delta x}{c_j}\left(\frac{1}{\psi_j} - 1\right), \quad (23)$$

where ψ_j is given by equation (10) with

$$\gamma = \frac{2\pi f_{max}\Delta x}{c_j}. \quad (24)$$

Therefore, for a fixed maximum error $\eta > 0$, we choose M_j as the minimum $M \geq M_{min}$ such that

$$\mu(M,c_j) \leq \eta. \quad (25)$$

Following the numerical example above, in Figure 3 we show the values of M_j according to equation (25) for different values of c_j and η.

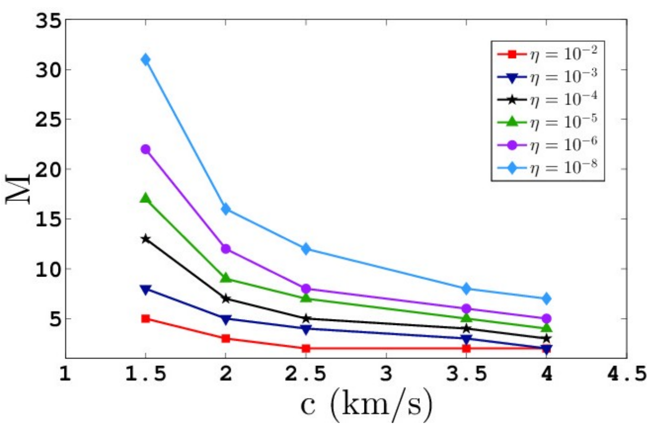

Figure 3: Values of the length operator M_j for different values of c_j and η.

Conclusions

The described approach finds differents values for the length of the spatial operator of a FD scheme for the unidimensional wave equation according to the velocity. When the velocity increases the length decreases. Then, we can expect that the computational cost for the FD recursion formula will be reduced when compared with FD schemes with fixed length operators.

Acknowledgments

This work was kindly supported by the Brazilian research agencies CAPES and CNPq, Brazil, and the sponsors of the Wave Inversion Technology (WIT) Consortium, Germany.

References

Bartolo, L. D., C. Dors, and W. J. Mansur, 2012, A new family of finite difference schemes to solve the heterogeneuos acoustic wave equation: Geophysics, **77**, T187–T199.

Dablain, M. A., 1986, The application of high-order differencing to the scalar wave equation: Geophysics, **51**, 54–56.

Finkelstein, B., and R. Kastner, 2007, Finite difference time domain dispersion reduction schemes: Computational Physics, **221**, 422–438.

Kelly, K. R., R. W. Ward, S. W. Treitel, and R. M. Alford, 1976, Synthetic seismogram: a finite difference approach: Geophysics, **41**, 2–27.

Liu, Y., and M. K. Sen, 2009, A new time-space domain high-order finite-difference method for the acoustic wave equation: Computational Physics, **228**, 8779–8806.

——, 2011, Finite-difference modeling with adaptive variable-length spatial operators: Geophysics, **76**, T79–T89.

Virieux, J., 1984, Sh wave propagation in heterogeneous media: velocity-stress finite difference method: Geophysics, **49**, 1933–1957.

——, 1986, P-sv wave propagation in heterogeneous media: velocity-stress finite difference method: Geophysics, **51**, 889–991.