A Fourier iterative method for solving axi-symmetric integral equations with application to induction logging

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Abstract

This paper presents a new iterative method for solving axi-symmetric integral equations. To test the accuracy and efficiency of the method, a relatively simple one-dimensional layered medium geometry is assumed. In this case, the problem reduces to sequences of one-dimensional convolutions using fast Fourier transforms along the axial direction z. The equation is renormalized such that the Neumann series iterations converge rapidly, even in the limit of high contrasts. The transforms are designed to mimic their continuous counterparts on semi-infinite intervals. Validation compares numerical and well-known analytical results for layered earth models.

Introduction

Integral equation solutions using finite-element or moment methods result in full matrices usually limiting such methods to cases where relative anomaly volumes are small. For linearized solutions, the convolutional form of the integral equation is often exploited by Fourier transforming the equation into the spatial frequency domain to perform the convolution between the Green's function and the volume current using FFT's (fast Fourier transforms). The integral equation naturally results in order 1 Fourier-Bessel transforms. These transforms however, are not directly amenable to Cartesian coordinate FFT computation.

A form of the integral equation this method is designed to solve is given by Howard (2014). This type of equation is used to model induction logging tool response in borehole geophysical applications (Chew (1991)). In order to realize the time savings fast Fourier transform operation count proportional to $N\log_2 N$, it is necessary to keep the computation in memory. For three-dimensional problems, work stations can provide this type of CPU memory resource (perhaps 128 to 256 GB). In the case of axial-symmetry, the integral equation for the electric field component $\mathrm{e}_\phi(\mathbf{x})$ is pseudo-scalar and is shown to take the form

$$\mathrm{e}_\phi(\mathbf{x}) = \mathrm{e}_{\phi,0}(\mathbf{x}) + \frac{1}{2\pi}\int g_1(\mathbf{x},\mathbf{x}')\, j(\mathbf{x}')\,\mathrm{d}^3x'\,, \qquad (1)$$

where the volume current $j(\mathbf{x})$ depends on the unknown electric field $\mathrm{e}_\phi(\mathbf{x})$ and the formation profile function $p(\mathrm{x})$, i.e.,

$$\begin{aligned} j(\mathbf{x}) &= p(\mathbf{x})\,\mathrm{e}_\phi(\mathbf{x})\,, \\ p(\mathbf{x}) &= k^2(\mathbf{x}) - k_b^2\,, \end{aligned} \qquad (2)$$

and the integration is over all space. The azimuthally symmetric Green's function in equation (2) for a loop source transmitter is given by

$$g_1(\mathbf{x},\mathbf{x}') = \int_0^{2\pi} \cos(\phi-\phi')\frac{e^{ik_bR}}{4\pi R}\,\mathrm{d}\phi'\,, \qquad (3)$$

where $R = |\mathbf{x}-\mathbf{x}'|$ and k_b is the intrinsic quasi-static electromagnetic background wavenumber in units of $[m^{-1}]$. The known background electric field $\mathrm{e}_{\phi,0}(\mathbf{x})$ is the solution to the homogeneous space problem with wavenumber $k_b = (i\omega\mu_0\sigma_b)^{1/2}$ with $\mathrm{Im}(\mathrm{k_b}) \geq 0$ for a time factor of $e^{-i\omega t}$. Here σ_b [S/m] is the earth model background conductivity, $\mu_0 = 4\pi\times10^{-7}$ [H/m] is the magnetic permeability of free space, $\omega = 2\pi f$, and f is the transmitter frequency in Hz. For this analysis we use the Green's function $g_1(\mathbf{x},\mathbf{x}')$ in the formof Howard (2015), p. 11

$$g_1(\mathbf{x},\mathbf{x}') = \int_{-\infty}^{\infty}\mathrm{d}K_z e^{iK_z(z-z')}\int_0^{\infty}\mathrm{d}K_\rho K_\rho\,\frac{J_1(K_\rho\rho)J_1(K_\rho\rho')}{K^2-k_b^2}\,. \qquad (4)$$

Note that the integral equation defined by equation (1) is a convolution of the Green's function and the volume current $j(\mathbf{x}')$. Substitution of representation (4) into integral equation (1) thus gives

$$\mathrm{E}(K_\rho,K_z) = \mathrm{E}_0(K_\rho,K_z) + \mathrm{G(K)\,J}(K_\rho,K_z)\,, \qquad (5)$$

where

$$\mathrm{G(K)} = 1/(K^2-k_b^2)\,, \qquad (6)$$

and $K^2 = K_\rho^2 + K_z^2$. The function $\mathrm{J}(K_\rho,K_z)$ is the transform of the volume current defined as

$$\mathrm{J}(K_\rho,K_z) = \int_{-\infty}^{\infty} e^{-iK_zz'}\mathrm{d}z'\int_0^{\infty}\rho\mathrm{J}_1(K_\rho\rho')\,j(\rho',z')\,\mathrm{d}\rho'\,, \qquad (7)$$

with analogous transform definitions for the background and total electric fields. In the one-dimensional layered

medium case where the profile varies only in the z direction ($p(\mathbf{x}) = p(z)$) results in the simplification

$$\mathrm{J}(K_\rho, K_z) = P(K_z) * E(K_\rho, K_z)\,. \tag{8}$$

An iterative solution to the integral equation (5) can be written in the form

$$\mathrm{E}^{(n+1)}(K_\rho, K_z) = \mathrm{E}_0(K_\rho, K_z) + \mathrm{G(K)}\left(P(K_z) * E^{(n)}(K_\rho, K_z)\right), \tag{9}$$

for $n = 0, 1, 2, \cdots$.The iteration is initialized with the background field, i.e. $E^{(0)}(K_\rho, K_z) = E_0(K_\rho, K_z)$,

where

$$\begin{aligned} \mathrm{E}^{(n)}(K_\rho, K_z) &= \int_{-\infty}^{\infty} e^{-iK_z z}\mathrm{d}z \int_0^\infty \rho \mathrm{J}_1(K_\rho \rho)\, e_\phi^{(n)}(\rho, z)\,\mathrm{d}\rho\,, \\ & \quad n = 0, 1, \cdots . \end{aligned} \tag{10}$$

It is interesting and gratifying to note that the form of integral equation given by (5) or (9) is the same as that for the scalar field 2D equation given by equations (7.11), (7.12), and (7.13) in Howard (2015). The first iteration begins with the background field, i.e., $\mathrm{E}^{(0)}(K_\rho, K_z) = \mathrm{E}_0(K_\rho, K_z)$ on the right-hand-side of equation (9).

To consider the numerical convergence of the iterative solution to integral equation (1), rewrite it in terms of the linear integral operator L, i.e.,

$$\mathrm{e}_\phi(\mathbf{x}) = \mathrm{e}_{\phi,0}(\mathbf{x}) + \mathrm{L}\,\mathrm{e}_\phi(\mathbf{x})\,. \tag{11}$$

Formally, this iterative solution, if it exists, takes the form

$$\begin{aligned} \mathrm{e}_\phi(\mathbf{x}) &= \mathrm{e}_{\phi,0}(\mathbf{x}) + \mathrm{L}\,\mathrm{e}_{\phi,0}(\mathbf{x}) + \mathrm{L}^2\,\mathrm{e}_{\phi,0}(\mathbf{x}) + \cdots, \\ &= \left[\mathrm{I} - \mathrm{L}\right]^{-1} \mathrm{e}_{\phi,0}(\mathbf{x})\,. \end{aligned} \tag{12}$$

A necessary condition for the convergence of the Neumann series solution given by equation (12) is that the norm of the operator be less than one, i.e.,

$$|\mathrm{L}| < 1\,. \tag{13}$$

The convergence depends on the profile function $p(\mathbf{x})$ and the singularity of the Green's function $g_1(\mathbf{x}, \mathbf{x}')$. Habashy (1993), by modifying the equation, softens the singularity in the Green's function, resulting in a simple method to enhance convergence. Their idea is to modify the defining equation (1) by adding and subtracting the term $\frac{1}{2\pi}\mathrm{e}_\phi(\mathbf{x}) \int g_1(\mathbf{x}, \mathbf{x}') p(\mathbf{x}')\,\mathrm{d}^3x'$ and thus obtain a form of the integral equation more amenable to Neumann iteration, e.g.,

$$\begin{aligned} \mathrm{e}_\phi(\mathbf{x}) &= n(\mathbf{x})\,\mathrm{e}_{\phi,0}(\mathbf{x}) + \\ & \quad \tfrac{n(\mathbf{x})}{2\pi} \int g_1(\mathbf{x}, \mathbf{x}')\, p(\mathbf{x}') \left(\mathrm{e}_\phi(\mathbf{x}) - \mathrm{e}_\phi(\mathbf{x}')\right) \mathrm{d}^3x'\,, \end{aligned} \tag{14}$$

where the re-normalization function is defined as

$$n(\mathbf{x}) = 1/\Big(1 - m(\mathbf{x})\Big)\,, \tag{15}$$

and where the associated normalization function is

$$m(\mathbf{x}) = \frac{1}{2\pi} \int g_1(\mathbf{x}, \mathbf{x}')\, p(\mathbf{x})\,\mathrm{d}^3x'\,. \tag{16}$$

Notice there are only three types of numerical operations in this type of iterative solution of the integral equation : addition, element-by-element multiplication ($.*$ in Matlab (2015) syntax) and a FFT based convolution. So for this type of solution to be useful, it is important that the most time consuming operation, namely convolution, be efficient.

In this exploratory analysis, let us consider a relatively simple case when the profile depends only on z. In this case, it is possible and easier to solve a sequence of independent, small, one-dimensional problems, rather than a much larger but equivalent two-dimensional one by using a hybrid method. Thus, take the transform of integral equation (5) with respect to z to get

$$\mathrm{E}(K_\rho, z) = \mathrm{E}_0(K_\rho, z) + \mathrm{G}(K_\rho, z) * \Big(p(z)\,\mathrm{E}(K_\rho, z)\Big)\,, \tag{17}$$

where from equation (6)

$$\mathrm{G}(K_\rho, z) = \frac{e^{-\gamma|z|}}{2\gamma}\,, \tag{18}$$

for

$$\gamma = (K_\rho^2 - k_b^2)^{1/2}, \quad \mathrm{Rl}(\gamma) \geq 0\,. \tag{19}$$

For an N_T turn loop source of radius a_T at axial position z_T, with N_T turns and current I_0, the background electric field $e_{\phi,0}(\mathbf{x})$ is

$$e_{\phi,0}(\mathbf{x}) = i\omega\mu_0 N_T I_0 a_T\, g_1(\mathbf{x}, \mathbf{x}_T)\,. \tag{20}$$

Using representation (4) leads to the hybrid form of the background field

$$E_0(K_\rho, z) = \frac{i\omega\mu_0 N_T I_0 a_T}{2} J_1(K_\rho a_T)\, \frac{e^{-\gamma|z - z_T|}}{\gamma}\,, \tag{21}$$

where γ is defined by equation (19). The renormalized iterative form of integral equation (17) analogous to equation (14) is

$$\begin{aligned} \mathrm{E}^{(n+1)}(K_\rho, z) &= \mathrm{N}(K_\rho, z)\mathrm{E}_0(K_\rho, z) + \mathrm{N}(K_\rho, z) \\ & \quad \Big(\mathrm{G}(K_\rho, z) * \big(p(z)\mathrm{E}^{(n)}(K_\rho, z)\big) \\ & \quad - \mathrm{E}^{(n)}(K_\rho, z)\mathrm{M}(K_\rho, z)\Big)\,, \end{aligned} \tag{22}$$

where

```
  Et      = E0;
  T1      = Nkrz.*E0;
  fm2D    = ones(ninpts,1)*prf1D;
  G1      = exp(-gamma*abs(zz))./(2*gamma*unitz);
  G1s     = fftshift(G1,2);
  Ets     = E0;
  tinyn   = 1.0e-5*nzpts;
  nit     = 1;            % iteration number
while max(rsid(:)) > tinyn && nit < nitmax
   T2  = ifft(G1s.*fft(fm2D.*Ets,[],2),[],2);
   Et  = T1 + Nkrz.*(T2 - Mkrz.*Ets);
   if nit == 1; Etb = Et; end  % Born approximation
   rsid(:,nit)= sum(abs((1-Et./Ets).^2),2);
   Ets =  Et;
   nit = nit + 1;
end; % while max(rsid(:)) > tinyn && nit < nitmax
rsid = rsid/nzpts;
```

Table 1: *Matlab code segment showing Neumann iteration and stopping criterion.*

$$\begin{aligned} \mathrm{N}(K_\rho,z) &= 1/(1-\mathrm{M}(K_\rho,z))\,, \\ \mathrm{M}(K_\rho,z) &= \mathrm{G}(K_\rho,z)*p(z)\,. \end{aligned} \tag{23}$$

The stopping criteria for the iterative method implied in equation (22) is that either the residual $\mathscr{R}$, defined as

$$\mathscr{R} = \sum_{n_z=1}^{N_z} |1-\mathrm{E}^{(n+1)}(K_\rho,n_z)/\mathrm{E}^{(n)}(K_\rho,n_z)|^2/N_z\,, \tag{24}$$

is less than a preset tolerance or the number of iterations n exceeds a preset upper bound $N_{\max}$. Beginning with the known starting point $E^{(0)}(K_\rho,z)=E_0(K_\rho,z)$, the idea is to iteratively solve the one-dimensional integral equation (22) for $E(K_\ell,z)$ for all radial wavenumber values N_r in parallel for all quadrature abscissa $K_\ell, \ell = 1,2,\cdots,N_r$ and then, compute the solution to the integral equation by numerical quadrature of the Fourier Bessel- transform, i.e.,

$$\mathrm{e}_\phi(\rho,z) = \sum_{\ell=1}^{N_r} J_1(K_\ell\rho)K_\ell E(K_\ell,z)w_\ell\,, \tag{25}$$

where w_ℓ is the quadrature weight coefficient associated with abscissa K_ℓ.

Unlike the case of linearized response function methods, as given for example in Howard (1986), this integral equation iterative method solves for the electromagnetic field at all points in the formation for a fixed transmitter location. To compute an induction log response, therefore, requires a solution for each transmitter location used to generate the log data. This additional computational overhead increases the importance of efficient computational methods.

To illustrate the simplicity and succinctness of the integral equation FFT iterative method, a short segment of Matlab code based upon the Neumann iteration scheme developed here, including a stopping criterion based upon small residual $\mathscr{R}$ as defined by equation (24) and iteration count is included in table 1.

Layered medium profile function $p(z)$

A staircase one-dimensional layered medium profile function $p(z)=k^2(z)-k_b^2$ for M regions with ordered bed-boundaries z_m, such that $z_m > z_n$ when $m > n$ and for $M > 2$ can be written in the form

$$p(z) = k_1^2\,u(z_1-z) + \sum_{m=2}^{M-1} \mathrm{rect}\Big(\frac{z-d_m^{(+)}}{2\,d_m^{(-)}}\Big)k_m^2 + u(z-z_M)\,k_M^2 - k_b^2\,, \tag{26}$$

where the step function $u(z)$ is defined as

$$u(z) = \begin{cases} 1, & z \ge 0, \\ 0, & \text{otherwise}, \end{cases} \tag{27}$$

and similarly the rectangle function $\mathrm{rect}(z)$ is defined as

$$\mathrm{rect}(z) = \begin{cases} 1, & |z| < 1/2, \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

The profile distances $d_m^{(\pm)}, m = 2,3,\cdots,M$ are given by

$$\begin{aligned} d_m^{(-)} &= \frac{z_m - z_{m-1}}{2}\,, \\ d_m^{(+)} &= \frac{z_m + z_{m-1}}{2}\,. \end{aligned} \tag{29}$$

Note that $p(z)$ has the piece-wise constant property

$$p(z) = k_m^2 - k_b^2\,, \text{ when } z_{m-1} \le z \le z_m\,. \tag{30}$$

Thus computing the associated profile spectrum $P(K_z)$ for profile given by equation (26) determines

$$\begin{aligned} P(K_z) &= 2\pi\Big[((k_1^2+k_M^2)/2 - k_b^2)\,\delta(K_z) \\ &\quad +2\sum_{m=2}^{M-1} d_m^{(-)} e^{-iK_z d_m^{(+)}}\,\mathrm{sinc}(K_z\,d_m^{(-)})\Big]\,, \end{aligned} \tag{31}$$

where

$$\mathrm{sinc}\,z = \frac{\sin z}{z}\,. \tag{32}$$

Thus a reasonable choice for the background intrinsic wavenumber k_b is the shoulder bed average

$$k_b^2 = \frac{k_1^2 + k_M^2}{2}\,. \tag{33}$$

Actually, equation (31) applies to an infinite interval. For discrete transforms over a finite interval $\mathscr{Z}$, a bed-weighted average over the discrete interval $\mathscr{Z} = f_r\mathscr{N}_z\delta_s$ is used where the skin depth $\delta_s = 1/\mathrm{real}(k_b)$ and f_r is an empirical unitless constant chosen to be $f_r = 0.012$. The bed weighted averaging method results in a quadratic equation for k_b.

Normalization function $m(\mathbf{x})$

For axi-symmetric 1D layered media, the normalization function $\mathrm{M}(K_\rho, z)$ is defined by equation (22) and is suitable for numerical evaluation.

$$\begin{aligned} \mathrm{M}(K_\rho, z) &= \mathrm{G}(K_\rho, z) * p(z), \\ &= \int_{-\infty}^{\infty} p(z') \frac{e^{-\gamma|z-z'|}}{2\gamma} \mathrm{d}z', \end{aligned} \tag{34}$$

for

$$\gamma = (K_\rho^2 - k_b^2)^{1/2}, \quad \mathrm{Rl}(\gamma) \geq 0. \tag{35}$$

Use of the normalization function in the form $\mathrm{M}(K_\rho, z)$ given by equation (34) is preferable for this problem because it is associated with a set of smaller one-dimensional problems.

Numerical examples

Figure 1 is the large dynamic-range one-dimensional 28 layer Oklahoma input formation conductivity profile $\sigma(z)$ [S/m] used in the simulations to follow. Table 2 defines input parameters for the simulation. Figure 2 compares the computation of equation (34) as computed by 256 point Gauss-Legendre quadrature with an FFT numerical convolution with respect to z. Figure 4 shows the convergence of the iterative method as a function of K_ρ after 1,2,5 and 9 iterations. In this example, the background conductivity $\sigma_b = 0.7 * \sigma_{b0}$ (see figure 5). Figure 5 is the mean with respect to K_ρ of $|\mathrm{N}(K_\rho, z)|$ showing the importance of choosing k_b correctly because the series does not converge where $|\mathrm{N}(K_\rho, z)|$ is too large. In these examples the array $\mathrm{E}(K_\rho, z)$ dimension is (48,1024), and for 10 Neumann iterations of the code, including the segment shown in Table 1, the execution time is 0.03 s on an older 2.4 Ghz T7500 workstation.

The normalization arrays $\mathrm{M}(K_\rho, z)$ and $\mathrm{N}(K_\rho, z)$ percent relative differences shown in Figure 3 have total

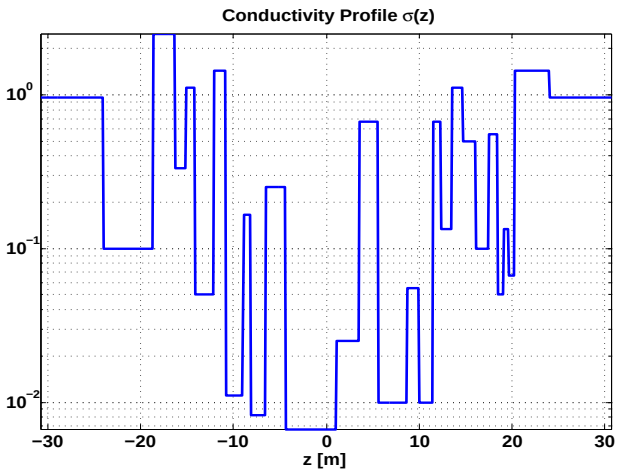

Figure 1: *Formation Conductivity Profile* $\sigma(z)[\mathrm{S/m}]$.

f	=	$20\,[\mathrm{kHz}]$,	σ_b	=	$0.5400\,[\mathrm{S/m}]$
N_T	=	10	I_0	=	$1\,[\mathrm{Amp}]$
a_T	=	$0.0508\,[\mathrm{m}]$	μ_0	=	$4\pi \times 10^{-7}\,[\mathrm{H/m}]$
Δz	=	$0.012\delta_s\,[\mathrm{m}]$,	N_z	=	1024
δ_s	=	$1/\mathrm{Im}(\mathrm{k_b})\,[\mathrm{m}]$,	L	=	$1\ [\mathrm{m}]$
ΔK_z	=	$\mathscr{K}_z/N_z$,	$\mathscr{K}_z$	=	$2\pi N_z/\mathscr{Z}$
$z(n)$	=	$z(1) + (n-1)\Delta z, \quad n = 1,2,$	$z(1)$	=	$-\mathscr{Z}/2$

Table 2: *Parameters used in numerical simulations.*

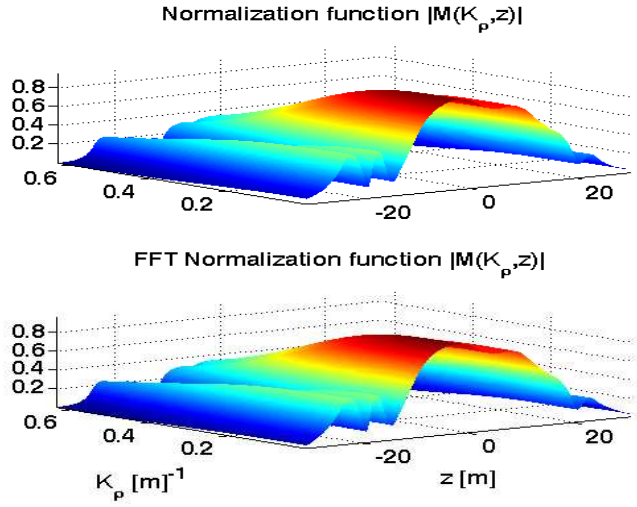

Figure 2: *Comparison of magnitudes of* $|M(K_\rho, z)|$ *as a function of* K_ρ *and* z.

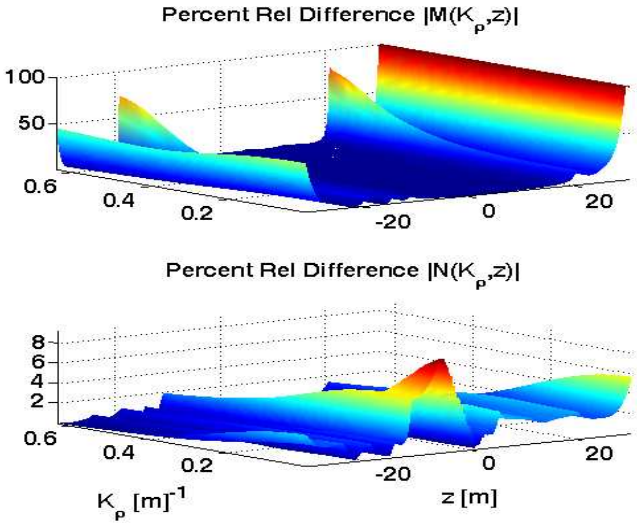

Figure 3: *Percent relative difference for* $\mathrm{M}(K_\rho, z)$ *and computation* $\mathrm{N}(K_\rho, z)$ *versus* K_ρ *and* z.

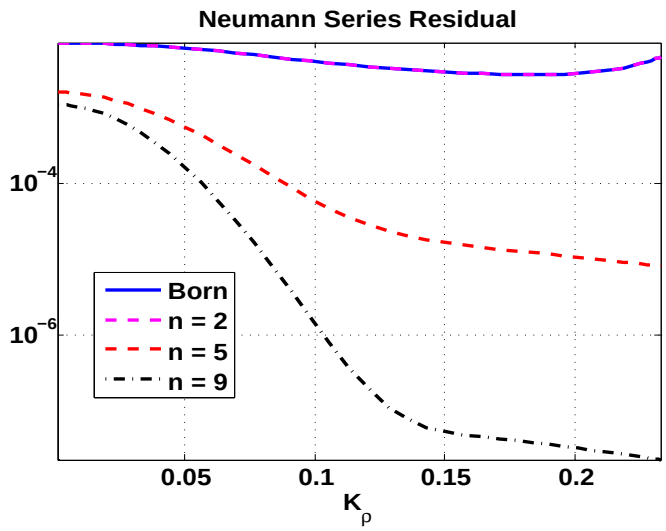

Figure 4: *Residual defined by equation (24) versus K_ρ after 2,5 and 9 iterations .*

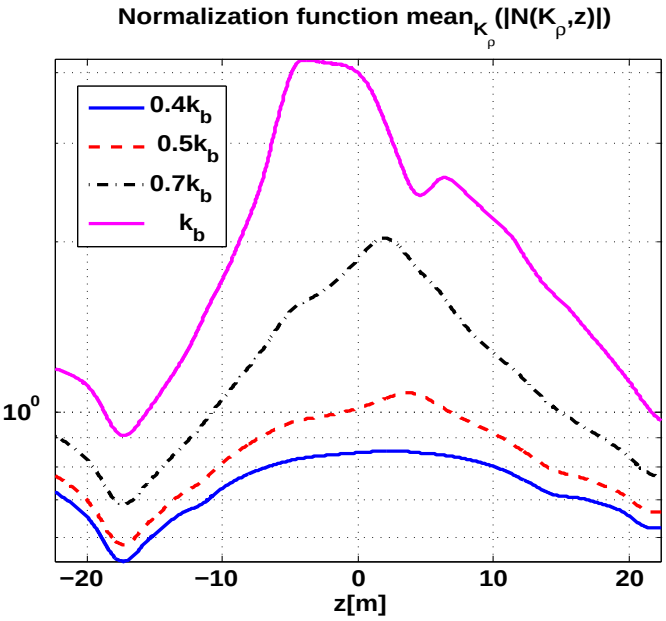

Figure 5: *Families of averaged normalization function for four choices of background intrinsic wavenumber k_b as shown in legend.*

range statistical values: $\text{mean(M)} = 7.32\%, \text{mean(N)} = 1.41\%, \text{median(M)} = 0.81\%,$ and $\text{median(N)} = 2.24\%$. If the extreme values of $|z|$ are excluded, because typically FFT results are known to have relatively large errors there, the mean value of M as seen by inspection of Figure 3 is then no more than a few percent.

Conclusions and discussion

This new Fourier iterative solution to axial symmetric integral equations is designed to be an efficient numerical method to compute synthetic induction logs because the computation consists only of FFT-based forward and inverse transforms and element-by-element matrix multiply $(*.*)$. The iterative procedure extends response function theory to the non-linear regime as required, for example, in high-contrast formations. Unlike the simpler response function methods, the iterative method solves for all observation points for a given receiver point. To insure convergence, the integral equation is renormalized. The renormalization places additional requirements on the choice of the background wavenumber k_b . The method is directly extend able to two-dimensional axial formations allowing for simultaneous invasion and layering. It is numerically efficient, and thus ideal as a forward model component in resistivity inversion methods. The ultimate goal is to apply the method to three-dimensional induction logging problems.

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