Optimized Finite Differences Scheme Applied to the Acoustic Wave Equation

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Abstract

The Finite Differences Method (FDM) is very popular in the acoustic wave modeling. However, the need to use of a very fine grid or a small sampling interval to improve the accuracy may increase the computational cost. An alternative employed in literature is to use optimized operators combined with the high FDM orders in order to infer a better precision without increasing the cost. Although, to obtain such efficiency, it is necessary to generate the dispersion and stability parameters for each discretization order, as they vary according to Finite Difference (FD) order and coefficients. We present a strategy to generate the dispersion and stabilities parameters used in the wave acoustic modeling using the high FDM orders. The appliance of this strategy combined with the optimized operators allows a more accurate modeling without altering the computational cost or reducing the cost with the same accuracy.

Introduction

In wave propagation in geological complex means, the seismic modeling is a tool that has been indispensable (Chu e Stoffa, 2012). The problem of the propagation of seismic waves simulation corresponds to solve differential equations describing the propagation inside the Earth.

Among the various numerical methods proposed to solve these differential equations, the FDM is the most popular, because it is easy to deploy and one of the most successful for being ideal for complex models, due to its efficiency (Alford *et al.*, 1974; Virieux, 1986, Chu and Stoffa 2012). This method is based on the approximation of the derivatives of the differential equations by replacing those by discrete approximations.

Generally, in order to approximate the derivatives, a truncated expansion of the Taylor series is employed, and then the coefficients relating to each point of the stencil of the formula of finite differences (operator) are obtained. Chu and Stoffa (2012) found two Binomial Window families that may be used to derive FD operators analytically. With a small alteration, this window may also be used to derive operators with greater dispersion property. However, the use of this window involves the control of a hard to determine parameter. Another fact is that this function needs to be handled carefully, because it may affect the final result in a substantial manner (Zhang and Yao, 2013).

A way to work around this problem is to use optimization techniques to generate optimized operators in order to minimize the error of dispersion (Chu and Soffa, 2012). Searching for this minimization, Zhang and Yao (2013) reduced the FDM numerical dispersions in the presence of high frequencies. The DF operator coefficients were optimized through the maximization of the convergence of the wavenumber, given an error limit.

Within this context, this work aims to present the precision obtained with the appliance of the optimized coefficients in the acoustic wave modeling with FDM. The dispersion and stability analysis that generated the parameters that optimized the acoustic wave modeling was also conducted.

The dispersion analysis and the numerical modeling demonstrated the new operator was more accurate than the conventional finite differences operator (Taylor) was, when the same number of points in the stencil of the operator is used in the calculation of the spatial derivatives.

Finite Differences Operator

The conventional FD operator for second spatial derivative for the $f(x)$ function may be written as a function of the truncation of the Taylor series around the $x = 0$ point in the following manner:

$$\frac{\partial^2 f}{\partial x^2} \approx \frac{1}{\Delta x^2}\sum_{n=-N/2}^{N/2} a_n \left[-\frac{2}{n^2}\cos(n\,\pi)\right] f(\Delta x)\ , \quad (1)$$

where Δx is the sampling interval in the x axis, N is the discretization order and a_n are the coefficients defined by the following binomial window (Chu and Stoffa, 2012):

$$a_n = \frac{\binom{N}{\frac{N}{2}+n}}{\binom{N}{\frac{N}{2}}} \quad (2)$$

A way to reduce the numerical dispersion is to adopt window functions to generate optimized coefficients for equation 1. Chu and Stoffa (2012) proposed a w_n window function to obtain optimized FD coefficients, which may be generated through the following manner:

$$\frac{\partial^2 f}{\partial x^2} \approx \frac{1}{\Delta x^2}\sum_{n=-N/2}^{N/2} w_n \left[-\frac{2}{n^2}\cos(n\,\pi)\right] f(\Delta x), \quad (3)$$

where w_n is generated from the following optimized window function

$$w_n = \frac{\binom{N+M}{\frac{N+M}{2}+n}}{\binom{N+M}{\frac{N+M}{2}}}\ , \quad (4)$$

where M is an even number, greater than zero and called an optimization parameter.

Using a final way for the optimized FD operator, one may combine the w_n window function with equation 3 in the following manner:

$$\frac{\partial^2 f}{\partial x^2} \approx \frac{1}{\Delta x^2}\sum_{n=-N/2}^{N/2} c_n\, f(\Delta x)\ , \quad (5)$$

where c_n is the final form of the coefficients to be used in the deployment, and may be defined as:

Conventional Coefficients - Taylor:

$$c_n = a_n\left[-\frac{2}{n^2}\cos(n\,\pi)\right] \quad n \neq 0 \quad (6a)$$

Optimized Coefficients - Binomial Window:

$$c_n = w_n\left[-\frac{2}{n^2}\cos(n\,\pi)\right] \; n \neq 0 \quad (6b)$$

For $n = 0$ we have:

$$c_0 = \sum_{n=-N/2}^{N/2} c_{-n} + c_n \quad (6c)$$

The M parameter of this window function is hard to determine, and, at the same time, needs to be handled carefully, because it may affect the final result significantly (ZHANG and YAO 2013). ZHANG and YAO (2013) determined the c_n coefficients using an optimization scheme that uses the maximization of the convergence of the wavenumber, given an error limitation. These coefficients are used in this work.

Absolute Error Analysis

Aiming to assess the improvement in the dispersion propriety, the spectral error analysis graph of the second derivative for various precision orders was constructed (Figure 1). Using equation 5, one could write the absolute spectral error formula, written as a function of the normalized wavenumber by the Nyquist wavenumber, that is:

$$Erro\left(\frac{K_x}{K_{xn}}\right) = -\left[c_0 + 2\sum_{n=1}^{N/2} c_n cos\left(\frac{k_x}{k_{xn}} n\pi\right)\right] - \left(\frac{k_x}{k_{xn}}\pi\right), \quad (7)$$

where $\frac{k_x}{k_{xn}}$ is the wavenumber normalized by the Nyquist wavenumber.

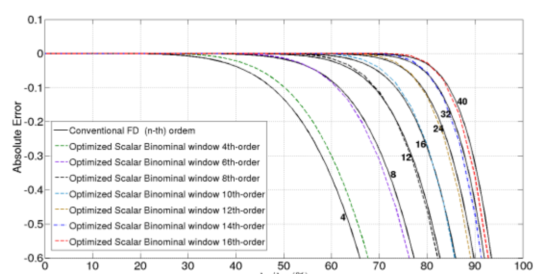

Figure 1 - Spectral Error Analysis for the Second Order Spatial Derivative Operator.

Wave Equation in Acoustic Mediums

The scalar wave equation considering means with constant density has following form:

$$\frac{\partial^2 P(x,z,t)}{\partial x^2} + \frac{\partial^2 P(x,z,t)}{\partial z^2} = \frac{1}{C(x,z)^2}\frac{\partial^2 P(x,z,t)}{\partial t^2} + f(t)\delta(x-x_f)\delta(z-z_f) \quad , \quad (8)$$

where $P(x,z,t)$ is the pressure field of the wave, x and z are the spatial coordinates, t is the time coordinate, $C(x,z)$ is the wave propagation velocity in the mean, $f(t)$ is the source term, δ is the Dirac Delta and x_f and z_f represent the application point of the seismic source in the x and z directions respectively.

Representing equation 8 in its discrete form, we shall have:

$$\frac{\partial^2 P(x,z,t)}{\partial x^2} = C_{i,j}^2\left[\frac{1}{\Delta x^2}\sum_{mi=-\frac{N}{2}}^{N/2} c_{mi}P_{i-mi}^n + \frac{1}{\Delta z^2}\sum_{mi=-\frac{N}{2}}^{N/2} c_{mi}P_{i-mi}^n\right] - f^n\delta(x-x_f)\delta(z-z_f), \quad (9)$$

where Δx^2, Δt^2,f^n are respectively the grid spacing in the x,z direction and temporal and $(x_f\ e\ z_f)$ represents the application point of the seismic source, which in this work is Ricker (Norman, 1953). The derivative regarding the time is usually discretized using the conventional second order FD operator. Incorporating this to equation 9 and using the optimized operator symmetry, one has the wave field explicitly as follows (Zhou and Greenhalgh, 1992):

$$P_{i,j}^{n+1} = C_{i,j}^2\Delta t^2\left[\left(\frac{1}{\Delta x^2}c_{mi=0} + \frac{1}{\Delta z^2}c_{mj=0}\right)P_{i,j}^n + \frac{1}{\Delta x^2}\sum_{mi=1}^{N/2} c_{mi}\left(P_{i-mi,j}^n + P_{i+mi,j}^n\right) + \frac{1}{\Delta z^2}\sum_{mj=1}^{N/2} c_{mi}\left(P_{i,j-mj}^n + P_{i+mi,j}^n\right)\right] + 2P_{i,j}^n - P_{i,j}^{n+1} - f^n\delta(x-x_f)\delta(z-z_f) \quad (10)$$

where n is the time step.

Dispersion and Stability Analysis

In the discretization of the wave equation through the FD operators, an error in the phase and group velocity will occur, because both start to depend on the grid spacing, the frequency of the signal and the angle of propagation. In the case of Wave Equation, this error appears on the form of numerical dispersion.

To determine the dispersion relation, one may consider the discrete expression of the propagation of a harmonic plane wave in an infinite and homogeneous mean:

$$p_{x,z}^t = e^{-i[k_x(x+ih)+k_z(x+jh)-\omega(t+n\Delta t)]}, \quad (11)$$

being ω the angular frequency, $i=\sqrt{-1}$ and k_x, k_z respectively the wavenumbers in the x and z directions, expressed by:

$$k_x = kcos\theta \quad \text{e} \quad k_z = ksen\theta, \quad (12)$$

where k is the wavenumber vector module and θ the angle between the vertical direction (z axis) and the direction of the propagation of the wave.

Replacing the equation 11 in the equation 10, and after a few algebraic manipulations, the following relation of the normalized phase velocity is obtained (Liu and Sen, 2009):

$$\frac{C_{FD}}{C} = \frac{2}{\mu kh}sen^{-1}\left\{\sqrt{\mu^2\sum_{m=1}^{N/2} c_m[A\ +\ B]}\right\} \quad (13)$$

wherein C is the velocity of the wave in the continuous mean, C_{FD} is the velocity of the phase of the wave on the discretized mean, c_m are the coefficient of the Stencil of the FD operator, h is the grid spacing, μ is the number of Courant - Friendrichs - Lewy (CFL), $A\ = sen^2\left(\frac{mkh\ sen\theta}{2}\right)\ e\ B\ =\ sen^2\left(\frac{mkhcos\theta}{2}\right)$.

The value of μ controls the stability of the FD scheme and may be defined by the analysis of the eigenvalues, where the s stability factor for the $2D$ wave acoustic modeling given by (Liu and Sen, 2009a):

$$\mu \le s = \frac{1}{\sqrt{2\sum_{m=1}^{N/2} c_m}} \quad (14)$$

being c_m the coefficients of the finite differences operators defined by equations 6a, 6b and 6c.

Fixating the value of μ, the equation 13 allows to assess the error in phase velocity of the discretized wave. A commonly used manner to determine the dispersion relating to the phase velocity consists in using the $\frac{1}{G}$ function, defined as:

$$\frac{1}{G} = \frac{h}{\lambda} = \frac{kh}{2\pi}, \quad (15)$$

The μ parameter is used to examine the dispersive nature of the waveform considering the phase velocity, which allows determining the lesser number of G wavelength points.

The equation (13) relates the normalized phase velocity with h grid space interval through the G variable. If the discretization process was analytical, the right side of the equation would be equal to 1.0 for all the h values. Any deviation of 1.0 is due to the error caused by the discretization of the wave equation. This error is responsible for the numerical dispersion.

The normalized phase velocity represented depends on the μ parameter, which has its maximum value estimated by the equation (14). However, when choosing such parameter, the curve of the normalized phase velocity must be generated within the 0.001 error limitation, in order to ensure the reduction of the presence of numerical dispersion in modeling.

Considering this restriction, in Figure 2 the behavior of the curve is within the interval only for $\mu = 0.14$, considering every angle below $\frac{\pi}{4}$. For $\mu = 0.2$, the curve was within the error limits only for $\theta \le \frac{\pi}{9}$. Thus, I recommend choosing $\mu = 0.14$ in order to ensure no significant dispersion occurs.

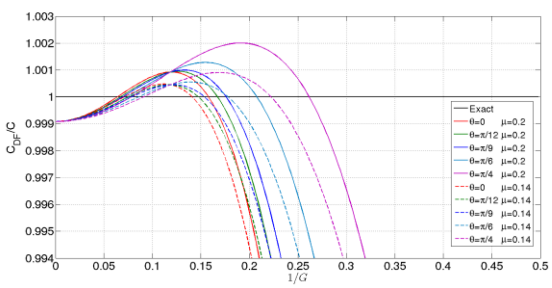

Figure 2 - Phase velocity assessment for 4th order optimized.

Using this strategy, one may obtain the following phase velocity analysis graph of the wave:

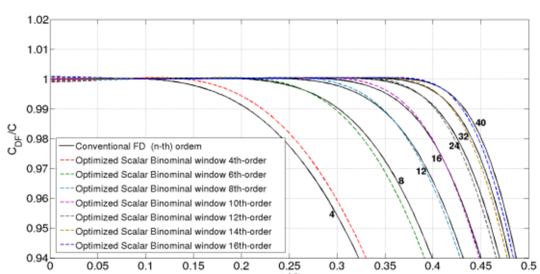

Figure 3 - Comparative phase velocity analysis using conventional and optimized FD coefficients.

Numerical Dispersion and Stability Criterion

To control the numerical dispersion in modeling, there is a relation between the lowest velocity of the C_{min} continuous means, the G parameter, which represents the number of points necessary to represent the smallest wavelength of the grid ($\boldsymbol{\lambda}_{min}$) and the cutoff frequency (f_{cutoff}). These parameters limit the maximum grid spacing value in order to not create excessive energy dispersion (Mufti, 1990). The dispersion relation is given by:

$$h \le \frac{\lambda_{min}}{G} = \frac{C_{min}}{G f_{cutoff}} \qquad (16)$$

where C_{min} is the lowest velocity of the mean, G is the number of points per wavelength and f_{cutoff} is the cutoff frequency.

The stability criterion is given by:

$$dt \le \frac{\mu\, h}{C_{max}} \qquad (17)$$

where dt is the time interval and C_{max} is the maximum velocity of the continuous mean.

Results

After the verification of the analytical precision of the operator, now its precision on the discrete setting will be assessed. In order for this to happen, initially the wave acoustic propagation will be simulated in a homogeneous mean, using the methodology adopted in the previous sections. The aim consists to validate the method and assess the precision between the orders of the optimized and conventional operator.

The propagation of seismic waves simulation via FDM involves the dispersion and stability relations to determine the h and dt parameters. Various works explain these relations for modeling with fourth order expansion with efficiency and precision. However, for higher orders these criteria are not well disseminated. In this work, the stability and dispersion parameters (Table 1) were generated based on the dispersion and stability criterion.

In order to validate the modeling code and assess the optimized FDM precision, a pressure field propagation simulation was conducted using the acoustic wave equation for the optimized 8th order and the 8th and 12th conventional orders (Figures 3 and 4).

This simulation was conducted on a homogeneous model of constant density with fixated dimensions of $x = 18{,}71\,Km$ and $x = 14{,}97\,Km$, means of velocity equal to $x = 1500\,Km/s$ and cut-off frequency of $30\,Hz$.

The result found in Figure 4 shows the optimized 8th order presents greater precision than the conventional 8th order, and in Figure 5 it is observed that the optimized 8th order presents a very close precision to the 12th conventional order.

N	G	μ	h (m)
C8	3,33	0,11	15
C12	2,94	0,08	17
C16	2,7	0,071	18
C24	2,5	0,061	20
C36	2,33	0,023	21
O8	2,9	0,07	17
O12	2,5	0,054	20
O16	2,3	0,048	22

Table 1 - Parameters for the acoustic wave modeling using conventional (C) and optimized operator (O).

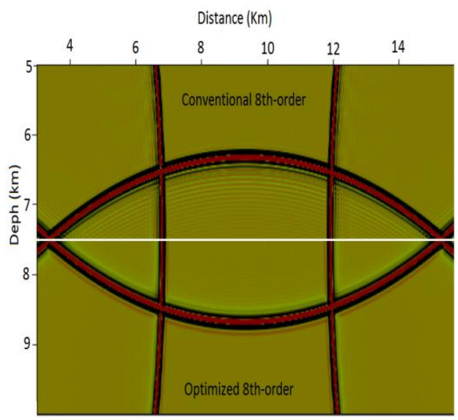

Figure 4 - Zoom of the Snapshot at 10.5 seconds

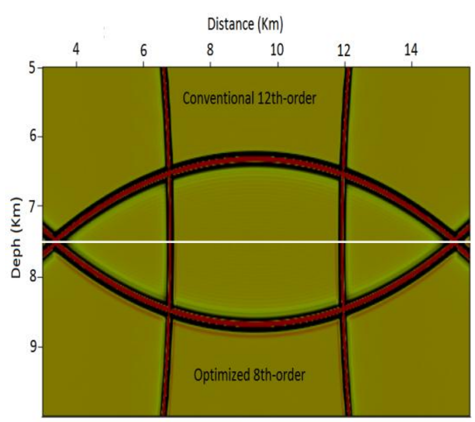

Figure 5 - Zoom of the snapshot at 10.5 seconds

Aiming to verify the optimized FDM precision in a more complex model, the modeling with optimized operator of the 8th and 16th order was assessed, in the modified Marmousi model (Figure 6), comparing with the different orders of the conventional model.

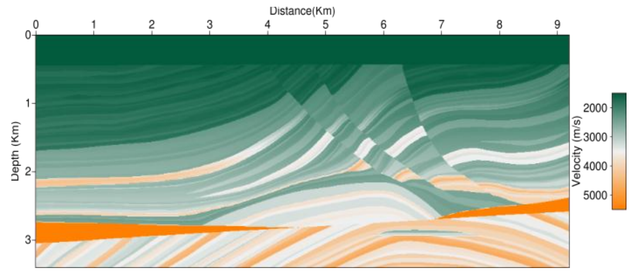

Figure 6 - Modified Marmousi model $h = 10,\ Np_x\ e\ Np_z$.

For purposes of comparison of precision, the wave field for the conventional thirty-sixth order was generated. For the traits (A, B and C) found in Figures (7 and 8), consider the continuous line represents the 36th order and the dashed is the order as indicated on the line.

The assessment in Figure 6 represents the record of the receiver located in the position (7000 m, 310 m). The source was inserted in position (4600 m, 310 m). The cutoff frequency used was of $60\ Hz$. The number of dots in the x direction is of $Np_x = 921$ and in the z direction $Np_z = 341$. The spacing between the points of the grid, for $G = 2.33$ is of 10 meters and the temporal sampling interval for $\mu = 0.07$ is equal to $0{,}00027$ s. The field generated by the conventional 8th order (dash A) obtained a deviation in relation to the conventional 36th order. By contrast, the 8th optimized order had almost the same wave field registered by the reference order, followed by the 12th conventional order. This indicates that for the same order and the same spacing between the grid, the optimized method presents a greater precision than the conventional method. These results reinforce the numerical analysis of the spectrum of the operator and the dispersion analysis.

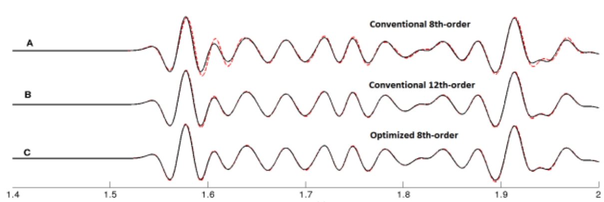

Figure 7 - Assessment of the precision between the 8th and the 12th conventional and the 8th optimized.

Assessing the 16th order in the Marmousi model, where the number of points is $Np_x = 614\ and\ Np_z = 227$. The receiver is situated in the position $(12{,}82\ Km, 462\ m)$. The record of intervals between $1{,}4s\ to\ 2s$ is seen on Figure 8. The spacing between the points of the grid, for $G = 2.0$ is of $12m$. The cutoff frequency used was of $60Hz$ and the temporal sampling interval for $\mu = 0{,}048$ is equal to $0{,}000104$ s. For dash **A** of the first interval, which corresponds to the conventional method, one can visualize a very significant deviation of the record of the 36th order. The record of the optimized method referring to dash **B** obtained practically the same result as the 36th conventional order.

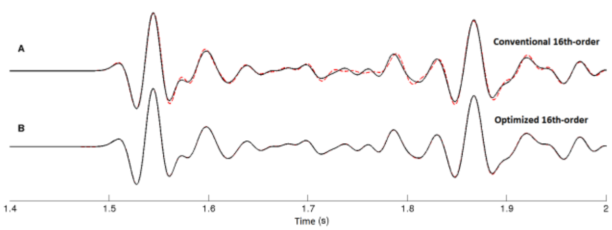

Figure 8 - Assessment of the precision between the 16th conventional and optimized

The results of the optimized method presented in both tests a better precision facing the conventional method, considering the same order. One must highlight that the gain of precision was obtained with the same computational cost. For precision proof purposes, the stability criteria regarding the optimized coefficients were used, considering the value of spacing between the grid increases and the temporal interval decreases in relation to the conventional method.

Conclusions

We presented the assessments and tests using the optimized FD scheme applied to the acoustic wave modeling. The precision of the optimized FD operators was assessed taking into account the error limit used by (Liu e Yao, 2013). We presented a way to generate the stability and dispersion parameters for any discretization order. Through the dispersion analysis, considering the CFL constant, the variation of the stability curve was limited within the 0.001 error limit to generate the number of points per wavelength. This strategy allowed generating the h and dt parameters for any discretization order without creating a significant dispersion and stability during the modeling.

Acknowledgments

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References

ALFORD, R. M., KELLY, K. R., BOORE, D. M., 1974, “Accuracy of finite difference modeling of the acoustic wave equation”, Geophysics, v. 39, n. 6, pp. 834–842.

CHU, C., STOFFA, P. L., 2012, “Determination of finite-difference weights using scaled binomial windows”, Geophysics, v. 77, pp. 57–67.

NORMAN, R., 1953, “The form and laws of propagation of seismic wavelets”, Geophysics, v. 18, n. 1, pp. 10-40.

MUFTI, I. R., 1990, “Large-scale three-dimensional seismic models and their interpretive significance”, Geophysics, v. 55, pp. 1166-1182.

LIU, Y., SEN, Y. M., 2009, “A new time-space domain high-order finite-difference method for the acoustic wave equation”, Journal of Computacional Physics, v. 228, pp. 8779–8806.

LIU, Y., 2013, “Globally optimal finite-difference schemes based on least squares”, Geophysics, v. 78, pp. T113–T132.

VIRIEUX, J., 1986, “SH-wave propagation in heterogeneous media: velocity-stress finite-difference method”, Geophysics, v. 51, pp. 889-901.

ZHANG, J. H., YAO, Z. X., 2013, “Optimized finite-difference operator for broad-band seismic wave modeling”, Geophysics, v. 78, pp. 1-6.