An approximate representation of the Fourier spectra of irregularly sampled functions

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Abstract

This paper describes an approximation for the one-dimensional Discrete Fourier Transform of Irregularly Sampled functions and its inverse. The discrete spectrum of one-dimensional discrete functions is then estimated within the range of validity of this approximation. The errors involved are estimated in a rather empirical way and some synthetic examples are shown. The extension of this approach for multidimensional functions' spectra estimation is straightforward and its applicability for higher dimension interpolation problems is outlined.

Introduction

Interpolation of multidimensional functions has gained much attention recently with applications in seismic data regularization for many different purposes. Many are the models used for mapping and reconstructing the acquired data with great differences in the quality of the interpolation achieved. From a long list of models, those with a more physical base tend to produce more acceptable results being Fourier mapping one of the most used approaches. However, given the irregular disposition of data, the Fourier spectrum evaluation problem is rarely a mathematically well posed one and virtually never counts on a established and efficient routine like Fast Fourier Transforms (*). Many are then the alternatives for handling the difficulties to implement a procedure that faces the trade off between the cost and required precision, efficiency and physical representativeness with Fourier sinusoidal basis. The most used approach formulates the Fourier mapping in the irregular case as an inverse problem:

$$d = \mathscr{F} D \quad \Longrightarrow \quad D = \mathscr{F}^{-1} d \tag{1}$$

where d, D, and $\mathscr{F}$, respectively, stand for the acquired data, its Fourier spectrum and a matrix with Fourier basis (complex exponentials) took at the irregular positions where d was acquired. That is, $\mathscr{F}$ plays the typical role of an irregular "inverse" discrete Fourier transform but, for the sake of simplicity, let's refer to $\mathscr{F}$ simply as a irregular Discrete Fourier transform (IDFT).

One of the major limitations on the Fourier approach for interpolation is the way it scales for multidimensional functions in relatively small problems. In seismic applications, an inverse for a 4D irregular discrete Fourier transform may demand unavailable resources of CPU and memory to compute in a systematic way.

Among with several initiatives to quantify the accuracy achieved in interpolation of irregularly sampled, band limited or not, unidimensional or multidimensional functions (see, for instance Eldar, 2006, and/or Guevara, et al., 2010), we can also find successful attempts to represent the Fourier spectrum of such functions (Duijndam et al., 1999) and derived practical applications with regular transforms, neglecting small departures from a regular embedded grid acquisition (Naghizadeh and Sacchi, 2010). The latter is well known as the Minimum Weighted Norm Interpolation method (MWNI).

This paper discusses an approximate expression for irregular Fourier transforms and their inverses, the conditions where these approximations hold and the order of the implied errors. It is then expected that such an approximation replaces the "exact" and expensive one allowing for a broader application of Fourier methods in interpolation and many other related fields. Motivation for the search for such an approximation is clearly the implied errors with MWNI method which is supposed to fail for acquisitions with a larger degree of irregularity. The scope here is limited to the very engine for interpolation and supposedly can offer new basis to address issues as cost and efficiency.

Due to limitations of this publication, the explanation here will be essentially limited to the 1D case. Higher dimension extensions will be indicated whenever appropriate. Also only the case where the inverse is supposed to exist will be addressed, postponing cases where the data have gaps and point sampling accumulation for another publication.

An approximation for Irregular Discrete Fourier Transforms

Discrete Fourier transforms (DFT) decompose a set of N measurements $d(x_n)$, $n = 0, N-1$, in a set of N sinusoidal functions with weights $D(k_m)$, $m = 0, N-1$. They are usually written as,

$$d(x_n) \;=\; \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} D(k_m)\, \mathrm{e}^{-2i\pi k_m x_n} \;. \tag{2}$$

DFTs usually deal with evenly spaced sets of x_n ($\Delta x = x_n - x_{n-1} = Const.$), and also regular sets of wave-numbers k_m, $m = 1, N$, with $\Delta x \Delta k = 1/N$. Under these conditions, DFTs are orthogonal transformations for which very efficient algorithms are available. Let's consider the case where measurement points departs from a regular grid by a small quantity: $\chi_n = x_n + \delta_n$, x_n an evenly spaced set of points. From (2) we can write,

$$\begin{aligned} d(x_n + \delta_n) &= \tfrac{1}{\sqrt{N}} \textstyle\sum_{m=0}^{N-1} D(k_m)\, \mathrm{e}^{-2i\pi k_m (x_n + \delta_n)} \\ &= \tfrac{1}{\sqrt{N}} \textstyle\sum_{m=0}^{N-1} D(k_m)\, \mathrm{e}^{-2i\pi k_m x_n}\, \mathrm{e}^{-2i\pi k_m \delta_n} \;. \end{aligned} \tag{3}$$

Now, since δ_n is small, we can approximate the exponential with δ_n as limited sum of terms in a Taylor series as

$$d(x_n+\delta_n) \approx \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} D(k_m)\,\mathrm{e}^{-2i\pi k_m x_n} \left[1-2i\pi\delta_n k_m - 2\pi^2\delta_n^2 k_m^2 + O(\delta_n)^2\right] \quad (4)$$

In expression (4) there is an embedded regular DFT. Let's denote it as $\mathbb{F}$ and rewrite (4) in matrix form as,

$$\mathbf{d} \approx \left\{\mathbb{F} - 2i\pi\boldsymbol{\delta}\mathbb{F}\mathbf{k} - 2\pi^2\boldsymbol{\delta}_n^2\mathbb{F}\mathbf{k}_m^2 + \ldots\right\}\mathbf{D}$$

where the series inside curly braces is a Taylor series for the irregular discrete Fourier transform:

$$\mathscr{F} = \sum_{j=0}^{\infty}\frac{(-2i\pi)^j}{j!}\boldsymbol{\delta}^j\mathbb{F}\mathbf{k}^j \ . \quad (5)$$

The interest in an expression like (5) lies in the opportunity to perform irregular DFTs efficiently (using FFTs), to analyze important features like invertibility and completeness and also, to estimate an inverse under a given degree of approximation. Matrices $\boldsymbol{\delta}$ and $\mathbf{k}$ are diagonal when there is no accumulation points and/or gaps and brings minor additional CPU cost to operate.

The way (5) was derived implies that convergence is guaranteed as long as Taylor expansion converge. A requirement for the fast convergence of the series is that $2\pi\delta_n k_m < 1$ for any n and m. This is usually assured since the greatest k is $1/2\Delta x$, the Nyquist wave-number for the embedded discrete Fourier transform, and if one chooses

$$\delta_n < \frac{\Delta x}{\pi},\ \forall\, n,\quad n = 0, N-1 \quad (6)$$

Figure 1 shows how singular values for different approximations compares to the exact $\mathscr{F}$ with greatest δ limited to $0.3\Delta x$. With the proper degree of zoom it is possible to see that the third degree approximation is already acceptable.

At this point, let's mention that higher dimension transforms can be treated much like the 1D case. A multidimensional $\mathscr{F}$ can be approximated via a multidimensional Taylor series with convergence assured by relations like (6).

The approximate estimate for the Fourier spectrum of irregularly sampled functions

Expression (5) allows one to estimate the inverse $\mathscr{F}^{-1}$ in an approximate way. Let's consider a least square inverse([1]) for $\mathscr{F}$,

$$\mathscr{F}^{-1} \Longrightarrow \left[\mathscr{F}^H\mathscr{F}\right]^{-1}\mathscr{F}^H \ . \quad (7)$$

Considering the series in (5) and given that $\mathbb{F}^H\mathbb{F} = \mathbb{I}$, the term inside brackets above takes the form,

$$\mathscr{F}^H\mathscr{F} = \mathbb{I} + 2i\pi\left(\mathbf{k}\mathbb{F}^H\boldsymbol{\delta} - \boldsymbol{\delta}\mathbb{F}\mathbf{k}\right) + O(\delta)^2 \ . \quad (8)$$

Hence, if (8) converges, one can write,

$$\left[\mathscr{F}^H\mathscr{F}\right]^{-1} = \mathbb{I} - 2i\pi\left(\mathbf{k}\mathbb{F}^H\boldsymbol{\delta} - \boldsymbol{\delta}\mathbb{F}\mathbf{k}\right) + O(\delta)^2 \quad (9)$$

and the inversion for $\mathscr{F}$ as in (6) as well as the spectrum D as in (1) are known up to a given degree of approximation on powers of δ.

Again, deriving multidimensional analogues for (8) and (9) is straightforward.

[1]When there are accumulation points and/or gaps the problem is likely to be ill posed.

A synthetic application - Seismic data regularization

Seismic data are usually acquired in a almost regular way. Departures from predetermined positions are kept as small as possible. However, for many different reasons, it may happen that trace locations do not correspond to the center of bins in a regular grid and/or gaps and accumulation bins are present. Seismic traces are sets of measurements taken at a regular time interval. Thus, interpolation is demanded only for spatially regularizing seismic surveys.

Seismic trace interpolation usually requires *a priori* information that can be provided in a easier way in the $f-k$ domain. Although this paper do not deal with ill posed problems where *a priori* information are a must, for compatibility purposes, let's keep the $f-k$ approach. In this case, given a set of irregularly acquired seismic data $d(\chi_n, t)$, where t is the time taken at regular intervals dt, one would like to estimate the $f-k$ spectrum $D(k, f)$, where f is the temporal frequency. A preliminary regular temporal DFT can take the acquired data to the space-frequency domain,

$$\tilde{d}(\chi_n, f) = \mathrm{DFT}_{tf}\, d(\chi_n, t)$$

and the estimate of $D(k, f)$ follows directly from the application of (1) as,

$$D(k, f) = \mathscr{F}^{-1}\tilde{d}(\chi_n, f) \ . \quad (10)$$

Since $\mathscr{F}^{-1}$ is known up to a given order of approximation (equations 5, 7, and 9), the estimate of $D(k, f)$ is immediate. A synthetic seismogram, irregularly sampled, is shown at figure 2. Trace positioning was made so as no trace departs from a regular grid by more than $0.3\ \Delta x$. A sixth order approximation for $\mathscr{F}^{-1}$ is free of errors greater than 0.003 in this example. There is just one trace for each of the grid points where an embedded DFT is defined.

The seismogram in figure 2 has 5 events: 4 linear and a circle. The circle is there for curvature probing purposes. Frequency content varies with dip so as to reduce (not eliminate) spatial aliasing. Amplitudes varies with frequency content so that all events have the same visibility. One of the 4 linear events is rather weak. Its small amplitude is especially interesting when additional, non linear constraints (*a priori* information) are imposed and has no purpose at this paper.

Summary and Conclusions

The possibility to approximately estimate the discrete Fourier spectra of irregularly sampled functions was demonstrated. Essentially a Taylor series expansion and truncation, the Fourier spectra is determined in a given degree of reliability which depends on how large measuring positions depart from an embedded regular grid. The approximation is a limited sum of regular DFTs which complexity is currently expressed as $MN\log N$ with M the number of DFTs used to reach an acceptable error and N is the dimension of the problem. As compared to the N^3 complexity to compute the inverse of irregular Fourier transforms, this approximation is expected to provide more usable basis for further developments in this field.

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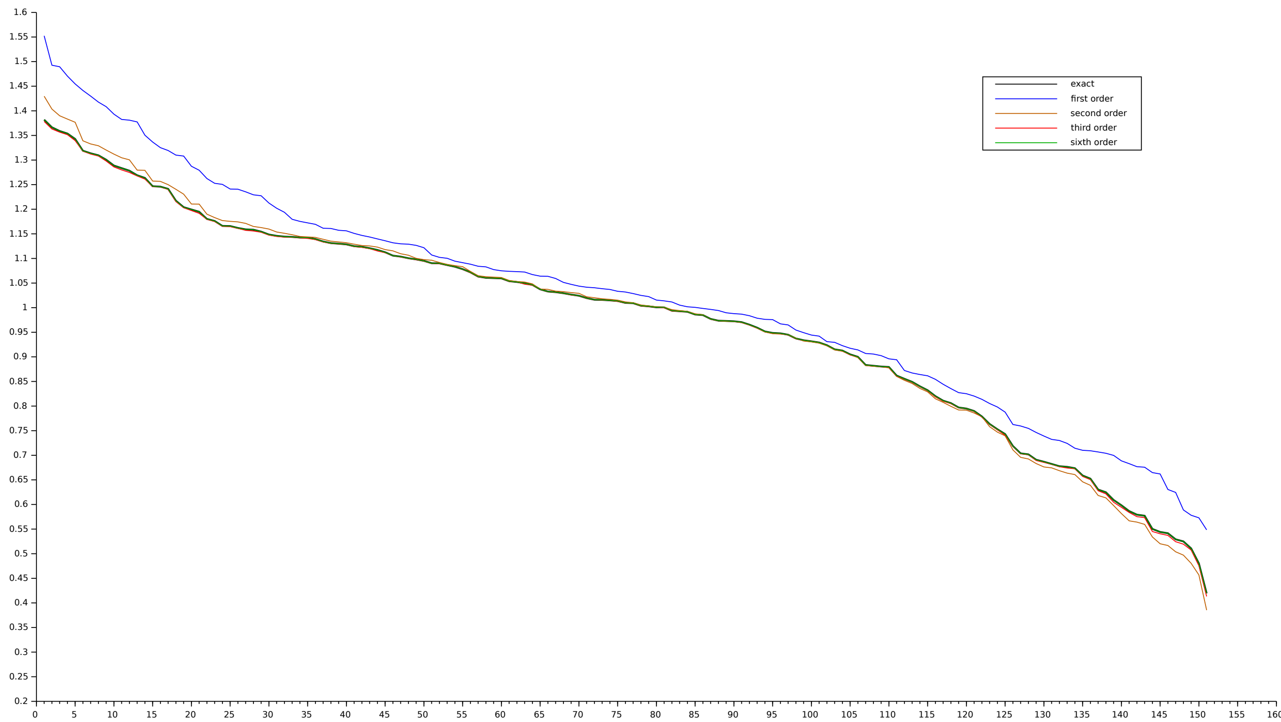

Figure 1: *A comparison between the singular values of the exact $\mathscr{F}$ and first, second, third, and sixth degree approximations. Recommended a 400% of zoom.*

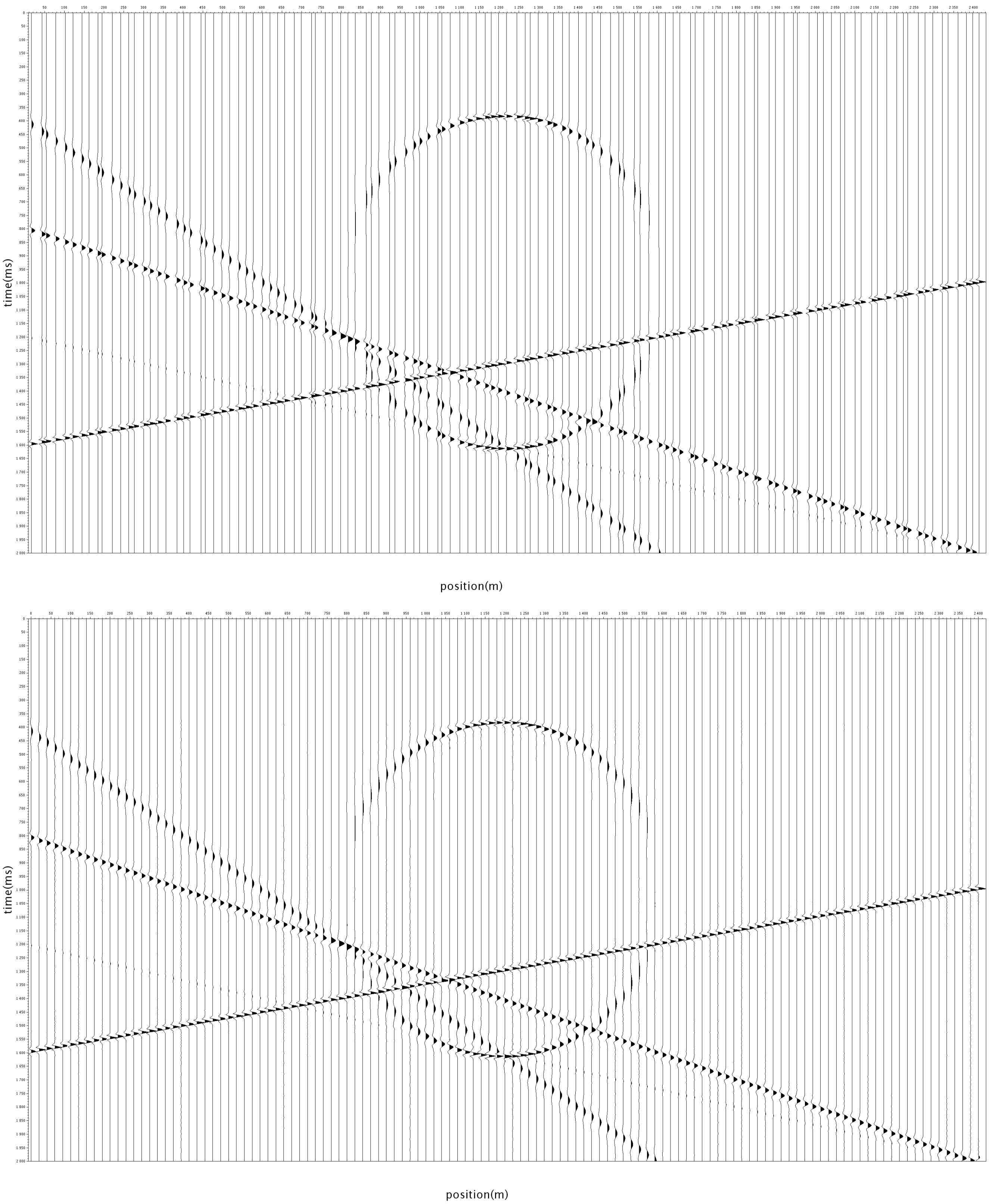

Figure 2: *Seismograms with 5 events: 4 linear and a circle. They are slightly in alias showing that interpolation is tolerant to a mild degree of aliasing. Top, irregularly sampled seismogram. Bottom, regularized seismogram.*

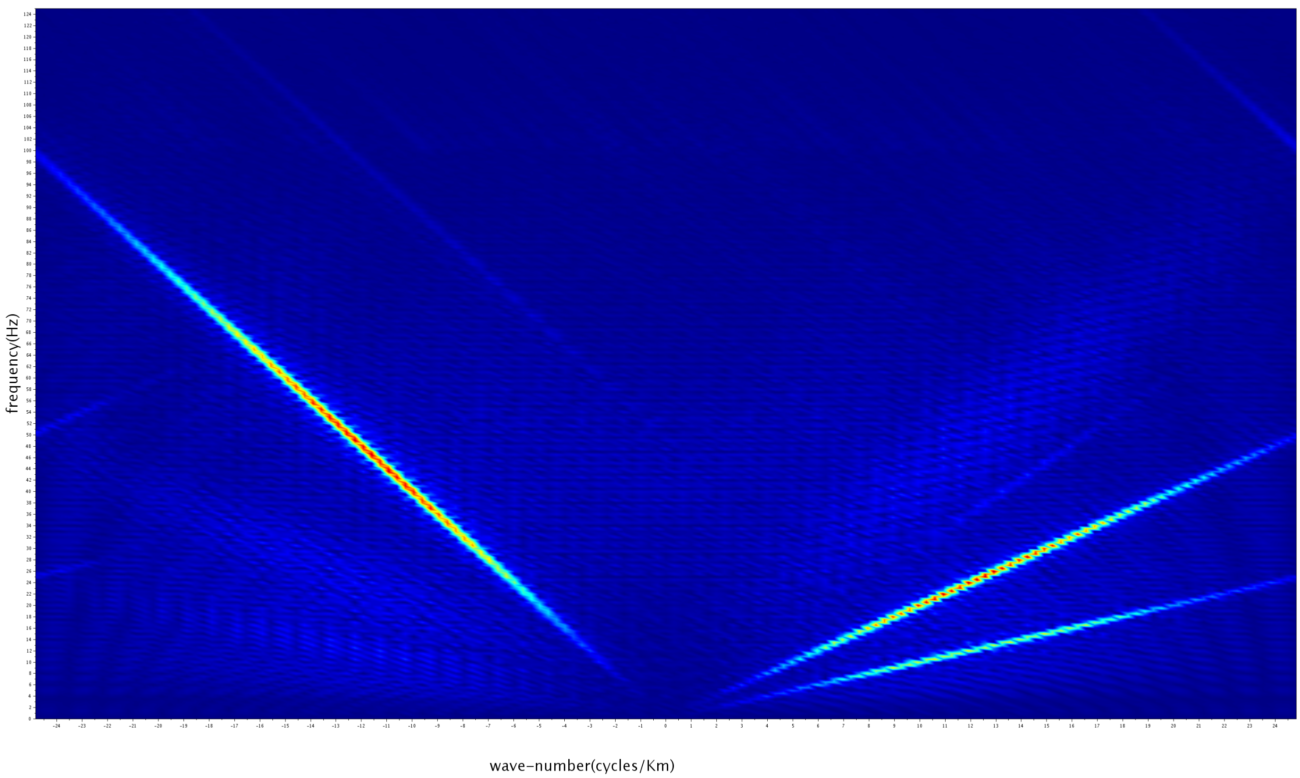

Figure 3: *The $f-k$ spectrum estimated with $\mathscr{F}^{-1}DFT_{tf}$. The circular event is dispersed since it corresponds to a linear combination of many "straight lines". The weak linear event demands special attention to be seen. Spatial aliasing is present. It can be also seen "straight strips of energy" with dips that are opposed to each of the 4 linear events. They belong to the boundary extensions (not shown in figure 2) used for reducing boundary artifacts.*