# Modeling of wave propagation in a two-dimensional heterogeneous elastic medium using the method of Differences Fintas

Andrea Lopez Duarte, Universidad de Pamplona - Colombia, Carlos Arturo Parra, Universidad de Pamplona - Colombia, Francisco Cabrera, Universidad de Pamplona - Colombia



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## Abstract

The propagation of waves are natural or artificial phenomena that are transmitted in elastic media, their propagation is mulled using dynamic elastic differential equations which have a temporal and a spatial component which must be solved numerically. Derivatives with respect to time and space are solved using a second-order approximation through finite-centered finite-difference operators. Because the modeling is in an isotropic environment the values of the spatial axis are positive in depth. The model is established by the use of speed and voltage in a discrete and staggered grid, which takes into account the deformation generated in the medium due to the voltage and the impact of the wave in it. This deformation effect will be analyzed mathematically taking into account the coefficients of lame, reflection and transmission of the wave to see the natural wave behavior. Given the fact that the modeling of the wave is computational, a treatment is made to the conditions of stability and dispersion so as not to obtain erroneous results and to be able to visualize the wave with a more real behavior.

Edge absorption methods were analyzed to avoid the visualization of non - existent reflections in the elastic medium.

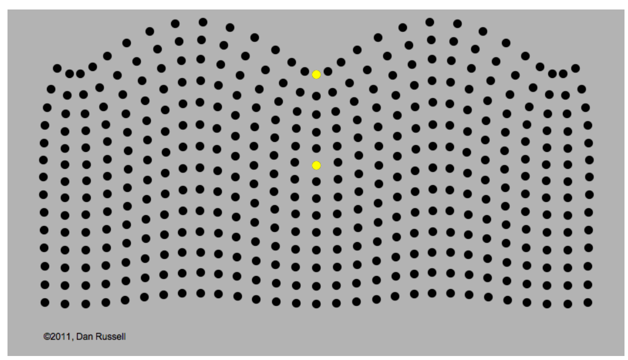
Key words: Finite Differences (DF), Elastic Medium, Tension, Deformation, Seismic Modeling

## Introduction

The source of excitation or energy pulse in an elastic medium does not generate an instantaneous deformation in the particles that constitute it but that said deformation requires time for it to be propagated in the different directions from its point of birth and passing through the different positions Of the disturbed medium. This phenomenon is well known for physical experiences and events that we see daily in our environment as waves on water surfaces, waves on a rope or the record of earthquakes or explosions, these are some of the examples that allow us to understand the Propagation of mechanical undulation movements.

The elements that form the medium undergo a deformation that allows the transmission of the perturbation (in this case in particular energy) from one point to another, in this way the wave advances through the medium (Image 1). In this process the disturbance must overcome the resistance that opposes the medium to be deformed, as well as the resistance to movement due to inertia. This energy transmission is carried out by the transmission of movement from one particle to another and not by a transmission of the medium as a whole.

The Finite Differences (DF) method is used traditionally to solve the equation of the acoustic and elastic waves, these equations describe the movement of the wave by the use of partial derivatives for both the temporal and the spatial part, Method allows the simulation of wave propagation in highly complex heterogeneous and geological elastic media. The most used DF operatives are centered second order. This method shows that it satisfies the stability condition and the numerical dispersion.




**Imag 1** deformation of the particles during transmission of a transverse wave (image taken from (Jonathan Lilly, n.) [6])

## Formulation

The formulation of the stepwise mesh scheme of Madariaga (1976) [1] where it expresses in terms of velocity of particles and tensions, the movement of the elastic wave equation using a circular expansion model in the elastic medium; And Virieux (1984) [2] which adapted the general scheme to model the waves in a 2D Cartesian system. These schemes allow greater ease for computational simulation.

For this Cartesian system the equations of motion for the velocity of the waves P and S are:

$$\rho \frac{\partial u_t}{\partial t} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} \qquad (1)$$

And

$$\rho\frac{\partial w_t}{\partial t}=\frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{zz}}{\partial z} \qquad (2)$$

And the constitutive laws of the isotropic environment are:

$$\tau_{xx}=(\lambda+2\mu)\frac{\partial u}{\partial x}+\lambda\frac{\partial w}{\partial z} \qquad (3)$$

$$\tau_{zz}=(\lambda+2\mu)\frac{\partial w}{\partial z}+\lambda\frac{\partial u}{\partial x} \qquad (4)$$

and

$$\tau_{zx}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \qquad (5)$$

Where u and w are the components of the displacement in x and z. U and w are the velocities of the particles, $\tau_{ij}$ they are the tensions, $\lambda$ and μ are the parameters of lame 'being μ the rigidity and p is the density.
The speed of comprehension is given by:

$$\alpha=\sqrt{\left[\frac{(\lambda+2\mu)}{\rho}\right]} \qquad (6)$$

And the shear rate by:

$$\beta=\sqrt{\left[\frac{(\mu)}{\rho}\right]} \qquad (7)$$

Taking the first derivative of time from the equations of the constitutive laws of the medium (Equations 1 to 5) and the substitution of velocity of the particles for displacement provide a first order system of the equations of velocity and stress that can be solved numerically. Levander (1988) [3]

**Implementation of Finite Differences**

The x, z and t coordinates are discretized by doing: x= (m±1)*h, z= (n±1)*h y t= ( $\ell$±1) Δt. h is the interval of the finite difference grid and t is the time sample in finite differences.

$$D_t^+u_t(m,n,\ell-1/2)= 1/\rho\,(m,n)\begin{bmatrix} D_x^-\tau_{xx}(m+1/2,n,\ell) \\ +D_z^-\tau_{xz}(m,n+1/2,\ell)\end{bmatrix} \qquad (8)$$

$$D_t^+w_t(m+1/2,n+1/2,\ell-1/2)= 1/\rho\,(m+1/2,n+1/2)\begin{bmatrix} D_x^+\tau_{xz}(m,n+1/2,\ell) \\ +D_z^+\tau_{zz}(m+1/2,n,\ell)\end{bmatrix} \qquad (9)$$

$$D_t^+\tau_{xx}(m+1/2,n,\ell)= \begin{bmatrix}\lambda(m+1/2,n) \\ +2\mu(m+1/2,n)\end{bmatrix}D_x^+u_t(m,n,\ell+1/2)+ \lambda(m+1/2,n)D_z^-w_t(m+1/2,n+1/2,\ell+1/2) \qquad (10)$$

$$D_t^+\tau_{zz}(m+1/2,n,\ell)= \begin{bmatrix}\lambda(m+1/2,n) \\ +2\mu(m+1/2,n)\end{bmatrix} D_z^-w_t(m+1/2,n+1/2,\ell+1/2)+ \lambda(m+1/2,n)D_x^+u_t(m,n,\ell+1/2) \qquad (11)$$

And

$$D_t^+\tau_{xz}(m+1/2,n,\ell)= \mu(m,n+1/2)\begin{bmatrix} D_z^+u_t(m,n,\ell+1/2) \\ +D_x^-w_t(m+1/2,n+1/2,\ell+1/2)\end{bmatrix} \qquad (12)$$

Where $D_t^+$ is the operator of difference with respect to time in the future position and, $D_x^\pm$ and $D_z^\pm$ are the differential operators with respect to space in the future position or the past. The positions in time are given by the operators m and n of the equations (from 8 to 14) where m + 1 is the future position, m-1 the past position and m the present position since the employed operator has the location of the position central.
The spatial derivative of the normal voltage used at the next moment of the velocity of the particles is given by:

$$D_x^+\tau_{xx}(m+1/2,n,\ell)={}^{c_1}\begin{bmatrix}\tau_{xx}(m+1/2,n,\ell) \\ -\tau_{xxt}(m-1/2,n,\ell)\end{bmatrix} \qquad (13)$$

Where C1 is the difference coefficient for the second order approximation of the first derivative.

**Stability and dispersion properties**

For a standard homogeneous medium the spectral analysis is performed at frequency and gives the following stability condition:

$$w_p\frac{\Delta t}{\Delta x}\sqrt{\frac{1}{\Delta x^2}+\frac{1}{\Delta z^2}}<1 \qquad (14)$$

Where wp is the velocity of the wave p. This stability condition is independent of the velocity of the S wave for the special case where = the stability condition is:

$$w_p\frac{\Delta t}{\Delta x}<\frac{1}{\sqrt{2}} \qquad (15)$$

For the case of n-dimensions, the root is summed from $\frac{1}{\Delta x_i^2}$ where i varies from 1 to n depending on the dimensions. In the case that all are equal, 2 is replaced by the number of dimensions in equation (15).
To see the dispersion is considered a plane wave with number of wave k, which forms an Angle with the x-axis. Given by:

$$\gamma=\sqrt{2}w_p\frac{\Delta t}{\Delta x} \qquad (16)$$

$$H = \frac{\Delta x}{\lambda} \quad (17)$$

Where equation 16 controls the dispersion number and 17 the number of nodes for the wavelength of the plane wave.

The non-dimensional wave results are given by:

$$q_p = \frac{\sqrt{2}}{\pi\gamma H}\sin^{-1}\left[\frac{\gamma}{\sqrt{2}}\sqrt{sin^2(\pi H\cos\theta) + sin^2(\pi H\sin\theta)}\right]$$

(18)

$$q_s = \frac{w_p\sqrt{2}}{w_s\pi\gamma H}\sin^{-1}\left[\frac{w_s\gamma}{w_p\sqrt{2}}\sqrt{sin^2(\pi H\cos\theta) + sin^2(\pi H\sin\theta)}\right]$$

(19)

Where $q_p$ is independent of the relation of Poisson's w. And $q_s$ depends on Poisson's ration through $\frac{w_s}{w_p}$ Virieux(1986) [2]

The values of H are small and approximately to a value of 0.1 when qp is approximately 1. (Fig 1.)

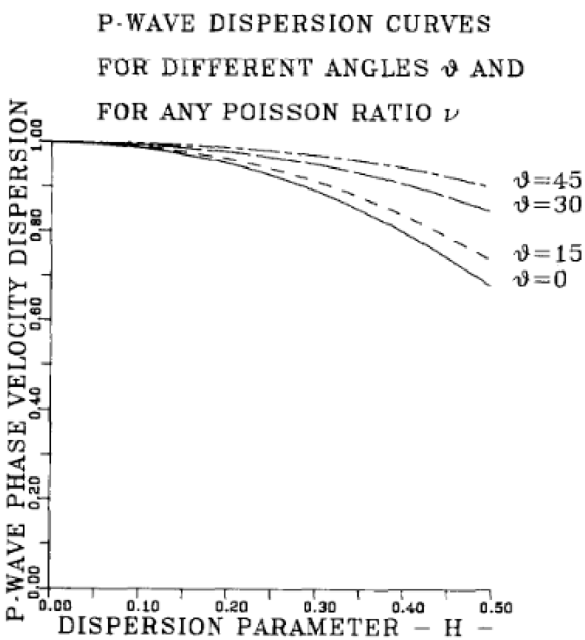



Fig. 1. Non-dimensional P wave velocity dispersion curves with a dispersion parameter = 0.8. Results for different angles of the plane wave with respect to the x axis are shown. These are independent of the relation of Poisson w. (Figure taken from virieux [2])

**Wave in the elastic medium**

The scheme shown in fig 2 and fig 3 by Virieux [2] explains the reflection in an elastic medium. Kelly et al [4] mainly observed this reflection because no radiation absorption conditions were applied

Starting from the source after the arrival of the direct P wave is very noticeable in the seismogram, the reflections of the PS wave is more noticeable in the horizontal component and the reflections of the PP wave are more notorious in the vertical component. Later in the horizontal component the diffraction of the ghost wave is created GSS and the other reflections in this component become diffuse, nevertheless in the vertical component they are notorious the reflections of the ghost waves GPP that soon they are interfered by the ghost waves GPS And GSP. Kelly et al. (1976) [4] mainly observed this reflection because no radiation absorption conditions were applied.

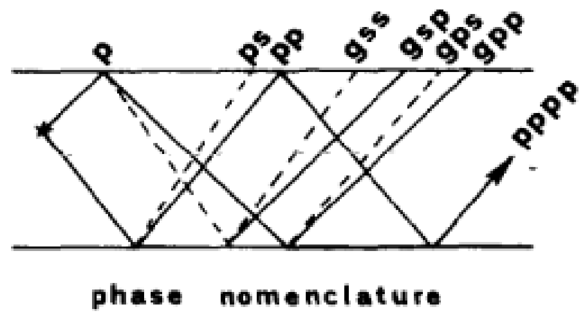



Fig. 2. Phase velocity dispersion curves of the non-dimensional P wave with a dispersion parameter = 0.8. Results for different angles of the plane wave with respect to the x axis are shown. These are independent of the relation of Poisson w. (Figure taken from virieux [2])

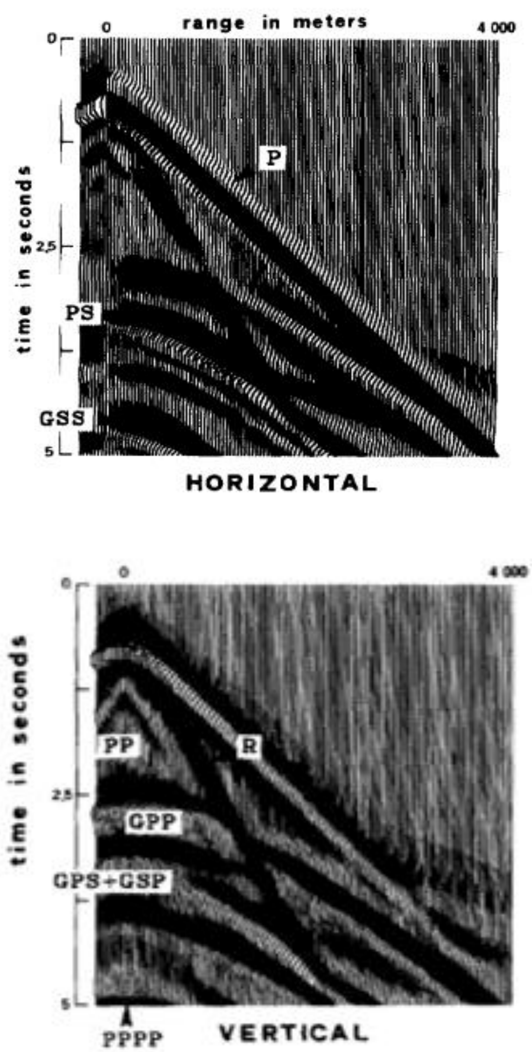



Fig. 3. Seismograms in which the various reflections can be observed in the upper part can be seen the horizontal

component and the reflections PS and GSS, in the lower part you can see the vertical component in which appear reflections PP, GPP , GPS and GSP.

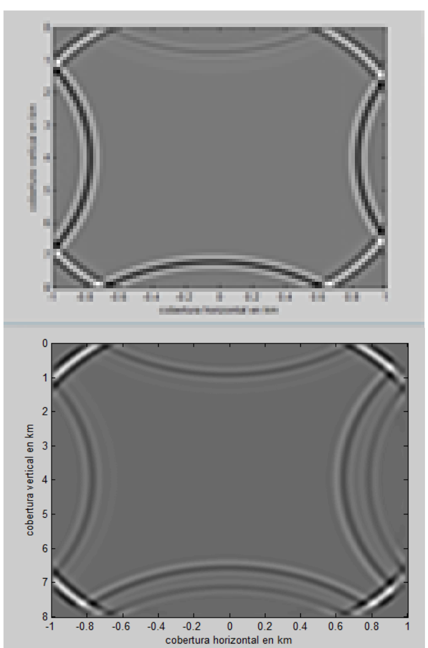
## Edge Absorption

At the moment of the wave propagation modeling using the DF method, a reflection problem is generated at the edges of the matrix used for the simulation, it should be in a real-life environment such as when an earthquake or an earthquake occurs Strong explosion the energy that is generated in such events generates a wave that continues its trajectory through the subsoil until it loses all its energy. Since the simulation is done computationally the area in which the wave transmits its energy is limited by the size of the mesh (matrix) that is being used, imaginary edges are generated which create non-existent reflections that translate into noises or data False In order for these reflections to be reduced, the scheme described by cerjan (1985) [5] has been implemented, and edges of 20 nodes wide are taken for each boundary of the mesh. At these edges is reduced slightly with each step of time (wave step at the selected edges). The reduction at each selected edge is gradually narrowed from zero at the inner boundary. The nodes between 1 and 20 of each edge are multiplied by:

$$G = e^{-[0.015*(20-i)]^2} \qquad (20)$$

Where i takes values between 1 and 20 that correspond to the node in which the wavefront is passing. G takes the value of 1 for i = 20 and a value of 0.92 for i = 1. This scheme can be used in the modeling with methods like Finite Elements and the Fourier method

a.




b.

Fig. 4. Comparison of the application of the scheme of reduction of the reflection of the wave at the edges. Part a shows the modeling of the wave without the attenuation and in part b is the attenuation of the absorption of the edges.

The simulation was performed taking as spatial components depth and distance (vertical and horizontal respectively), with this edge absorption allows the visualization of the reflections of the waves in elastic media can be easily differentiated from the computationally generated reflections and not Alter the reflections in non-homogeneous media such as the cases of two or more layers.

## Conclusions

The stability and dispersion conditions have been verified allowing it to be correct for the model implemented with the Finite Differences method. This scheme allows to create an approximate simulation to the reality of the transference of the wave in an elastic medium.
The vertical component of the seismogram allows a greater visualization of the reflections of the wave when transmitted in an elastic medium. Due to the behavior of the wave and the various reflections that is presented makes this simulation somewhat complicated to understand or visualize clearly which is why it is very necessary to use schemes that reduce the reflections of the edges as this would complicate the analysis much more Of the seismogram.
The reflection attenuation scheme at the edges greatly reduces the wave reflections at the edges of the mesh even though it does not eliminate those reflections completely and due to the simulation of the geophones in the upper part of the mesh is not of A lot of help on that edge.

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