Poroelastic modeling in stratified media: Biot-JKD equations

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Abstract

We represent an exact mathematical procedure for the analysis of the elastic wave propagation in plane layered porous media taking into account the effect of high temporal frequencies. For the characterization of the effect we use the dynamic permeability expression proposed by Jonhson, Koplik and Dashen in 1987. The algorithm is based on a formalism introduced by Ursin in 1983.

Introduction

In 1956, Biot published two famous articles describing the propagation of elastic waves in porous media containing a fluid, see Biot (1956a, 1956b). One of the important results of the poroelasticity theory is the identification of three waves: two compressional (fast and slow) and one shear, these are similar to the usual compressional and shear waves in an elastic medium, respectively. The slow compressional wave, also known as the slow wave of Biot, was experimentally confirmed by Plona (1980).

For high frequencies Biot presented a particular expression for two types of pore geometry: two-dimensional flow between parallel walls and three-dimensional flow in a circular duct, Biot (1956b). In 1987 Johnson, Koplik and Dashen published a general expression for the dynamic permeability in the case of random pores, leading to the Biot-JKD model. In this model, viscous stresses depend on the square root of the temporal frequency and only a non-dimensional physical parameter was involved. A generalization of this theory was given by Pride et al. (2004).

A numerical approach to solving equations directly in the time domain was proposed by Masson and Pride (2010). This work consists of a simple discretization of the fractional derivatives, defined by a convolution product. Another approach, based on the diffuse representation of the fractional derivative, was proposed by Hanyga and Lu (2005). More recently, a time-domain numerical modeling of Biot poroelastic waves was proposed in Blanc (2014). The method is based on a diffusive representation and replacing the convolution kernel by a finite number of memory variables that satisfy local-in-time ordinary differential equations, resulting in the Biot-DA (diffusive approximation) model.

Our procedure is based on a formalism introduced by Ursin (1983), who showed how Maxwell's equations, the equations of acoustics and the equations of isotropic elasticity all have a similar mathematical structure in an appropriate way.

In this paper, we add the equations of poroelasticity (higher frequency range) to Ursin's list. We develop Ursin's formalism for the case of a stack of homogeneous layers, i.e., when the material parameters are piecewise functions of the depth only. In this case many quantities can be computed with explicit algebraic formulas.

For the low-frequency range a similar algorithm was developed by Azeredo (2013).

Statement of the Problem

We shall consider wave propagation in a porous half-space $R=\bigcup_{k=1}^{k=N} R_k$, composed with stratified layers $R_k=\left\{x=(x_1,x_2,z)\in\mathbb{R}^3 : z_k<z<z_{k+1}\right\}$, where $0=z_0<z_1<\cdots<z_{N+1}=\infty$. Let $u=(u_1,u_2,u_3)$ and $w=(w_1,w_2,w_3)$ be the solid and relative fluid displacements, respectively. The Biot-JKD equations (higher-frequency case) in the time frequency (ω) domain, at each point $x\in R$, are (time dependence of $e^{-i\omega t}$ is assumed)

$$\begin{aligned}
&-i\omega\left(\rho v+\rho_f q\right)=\nabla\cdot\tau+F\\
&-i\omega\left(\rho_f v+\rho_w q\right)=-\nabla p-Dq+f\\
&-i\omega\tau=\left(\lambda_c\nabla\cdot v+C\nabla\cdot q\right)I+G\left(\nabla v+\nabla v^T\right)\\
&i\omega p=C\nabla\cdot u+M\nabla\cdot q
\end{aligned}\tag{1}$$

where

$$\rho=\varphi\rho_f+(1-\varphi)\rho_s,\rho_w=F_e\rho_f,D=\frac{\eta}{\kappa_0}\sqrt{1-i\frac{\omega}{\Omega}}$$

Here $v=-i\omega u, q=-i\omega w$ are the solid and relative fluid velocities, $F=(F_1,F_2,F_3)$, $f=(f_1,f_2,f_3)$ are the forces imposed on the solid and on the pore fluid, respectively, τ is the stress tensor, p is the pressure in the pore fluid, λ_c,G the Lamé coefficients, C,M the Biot moduli, ρ_f the density of the pore fluid, ρ_s the density of the elastic sceleton, F_e is the electrical formation factor, $0<\varphi<1$ is the porosity, κ_0 is the steady-flow (zero frequency) limit of the permeability, η the pore fluid viscosity, Ω is the circular frequency at which viscous boundary layers first develop, and I is the identity matrix. We assume that all material parameters are represented by piece-wise constant functions depended only the depth coordinate z, with the discontinuities at the points

$z = z_k, k = 1,2,\ldots,N$. At the discontinuity points z_k we suppose that the following functions are continuous:

$$[p] = [q]\cdot n = 0, [v] = [\tau]\cdot n = 0,\ n = (0,0,1)^T \quad (2)$$

The free boundary conditions are

$$z = 0:\ p = 0, \tau\cdot n = 0 \quad (3)$$

And finally, at the infinity the solution satisfies the following radiation conditions:

$$\lim_{|x|\to\infty}(v,q) = 0 \quad (4)$$

Method

1. Ursin's format. Consider the Fourier transforms in the two coordinates x_1, x_2

$$\hat{X}(k_1,k_2,z) = F_{x_1x_2}(X) \equiv \iint_{R^2} e^{-i(k_1x_1+k_2x_2)} X(x_1,x_2,z)\,dx_1\,dx_2$$

Let $(k_1,k_2)^T$ be the horizontal wavenumber and $k = \sqrt{k_1^2+k_2^2}, \gamma = k\omega^{-1}$. Applying the Fourier transform to (1) we obtain the EDO's system represented in the terms of $\hat{f},\hat{g},\hat{v},\hat{q},\hat{\tau},\hat{p}$.

Let

$$\Omega = k^{-1}\begin{pmatrix} k_1 & k_2 & 0 \\ -k_2 & k_1 & 0 \\ 0 & 0 & k \end{pmatrix}$$

The EDO's obtained can be simplified if we define

$$\tilde{x} = \Omega x, \tilde{v} = \Omega\hat{v}, \tilde{q} = \Omega\hat{q}, \tilde{\tau} = \Omega\hat{\tau}\Omega^T, \tilde{f} = \Omega\hat{f}, \tilde{g} = \Omega\hat{g}, \tilde{p} = \hat{p}$$

A straightforward calculation uncouples this system

$$\frac{d\Phi^{(m)}}{dz} = -i\omega M^{(m)}\Phi^{(m)} + S^{(m)}, m = 1,2 \quad (5)$$

where $\Phi^{(m)}$ are the $2n_m$- vectors ($n_1 = 3, n_2 = 1$),

defined as

$$\Phi^{(1)} = (\tilde{v}_3, \tilde{\tau}_{13}, -\tilde{q}_3, \tilde{\tau}_{33}, \tilde{v}_1, \tilde{p})^T, \Phi^{(2)} = (\tilde{v}_2, \tilde{\tau}_{23})^T$$

$S^{(m)}$ are the source $2n_m$-vectors, and $M^{(m)}$ are the $2n_m \times 2n_m$-matrices

$$M^{(m)} = \begin{pmatrix} 0 & M_1^{(m)} \\ M_2^{(m)} & 0 \end{pmatrix} \quad (6)$$

with symmetric $n_m \times n_m$-matrices $M_1^{(m)}, M_2^{(m)}$ and the submatrices and the corresponding source vectors giving in Appendix.

2. Diagonalization. Let's give briefly a derivation of the diagonalization procedure. We consider matrices of the form (6), where for simplicity we drop the superscript $^{(m)}$.

Assume that M_1M_2 has n distinct nonzero eigenvalues λ_j^2, $j = 1,2,\ldots,n$, with associated eigenvectors a_j, $j = 1,2,\ldots,n$, such that $a_j^T M_2 a_j = \lambda_j$. Here $\lambda_j = \sqrt{\lambda_j^2}$ with the branch chosen so that $\mathrm{Im}(\lambda_j) \ge 0$ and $\lambda_j > 0$ is real if λ_j is real. Define $b_j = \lambda_j^{-1}M_2a_j$. This vector is an eigenvector of M_2M_1 with eigenvalue λ_j^2. Using symmetricity of M_1, M_2 we obtain $a_j^T b_i = \delta_j^i$, where δ_j^i is the Kronecker delta.

Let L_1 be the $n\times n$ matrix whose j-th column is a_j, and let L_2 be the $n\times n$ matrix whose i-th column is b_i, then $L_1^{-1} = L_2^T, L_2^{-1} = L_1^T$. Introduce

$$\Lambda = \mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$$

Then $L_2\Lambda = M_2L_1$ and $M_1L_2 = L_1\Lambda$, which implies

$$M_1 = L_1\Lambda L_1^T, M_2 = L_2\Lambda L_2^T \quad (7)$$

Introducing the diagonal matrix $\tilde{\Lambda} = \mathrm{diag}(\Lambda,-\Lambda)$ and using (7), we finally obtain

$$M = L\tilde{\Lambda}L^{-1} \quad (8)$$

where

$$L = \frac{1}{\sqrt{2}}\begin{pmatrix} L_1 & L_1 \\ L_2 & -L_2 \end{pmatrix}, L^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} L_2^T & L_1^T \\ L_2^T & -L_1^T \end{pmatrix}$$

The explicit formulas for λ_j, a_j, b_j for Systems 1 and 2 are given in Appendix.

3. Reflection and transmission matrices. Firstly, we consider a homogeneous, source-free region of space. Dropping $^{(m)}$ we have a $2n$-dimensional system of the form (5) with M constant and $S = 0$. Let

$$\Phi = L\Psi \text{ and } \Psi = (U,D)^T \quad (9)$$

where U,D are n-vectors, characterizing up-going (U) and down-going (D) waves. Then

$$\Psi(z) = \left(e^{-i\omega\Lambda(z-z_0)}U(z_0), e^{i\omega\Lambda(z-z_0)}D(z_0)\right)^T \quad (10)$$

where z_0 is a fixed point in the same source-free region.

Consider an interface at z, where the material parameters vary discontinuously across z. We denote by $^{\pm}$ quantities evaluated at $z^{\pm} = z \pm 0$. Since Φ is continuous across z, we obtain

$$\Psi^{\pm} = J^{\pm 1}\Psi^{\mp} \tag{11}$$

where the jump matrix is

$$J = (L^{+})^{-1}L^{-} \equiv \begin{pmatrix} J_A & J_B \\ J_B & J_A \end{pmatrix}, J^{-1} = \begin{pmatrix} J_A^T & -J_B^T \\ -J_B^T & J_A^T \end{pmatrix}$$

and J_A, J_B are the $n \times n$-matrices

$$J_A = \frac{1}{2}\left[\left(L_2^{+}\right)^T L_1^{-} + \left(L_1^{+}\right)^T L_2^{-}\right], J_B = \frac{1}{2}\left[\left(L_2^{+}\right)^T L_1^{-} - \left(L_1^{+}\right)^T L_2^{-}\right]$$

Next, we consider a stack of layers $0 < z_1 < \ldots < z_N < \infty$. We have $\left(U_N^-, D_N^-\right)^T = J_N^{-1}\left(0, D_N^+\right)^T$, where we have used that there is no up-going wave below the last interface at $z = z_N$. So, we obtain

$$U_N^- = \Gamma_N D_N^-, D_N^+ = T_N D_N^- \tag{12}$$

where

$$\Gamma_N = -\left(J_{B,N}^T\right)\left(J_{A,N}^T\right)^{-1}, T_N = \left(J_{A,N}^T\right)^{-1} \tag{13}$$

Here Γ_N is the reflection matrix and T_N is the transmission matrix from the last interface $z = z_N$, respectively. Let $j < N$ and $\Delta z_j = z_{j+1} - z_j$, $j = 1, 2, \ldots, N-1$, is the layer thickness. Then by jumping across the layer boundary and using (10), (11) we obtain

$$\begin{aligned} U_j^- &= J_{A,j}^T e^{i\omega\Lambda_j \Delta z_j} U_{j+1}^- - J_{B,j}^T e^{-i\omega\Lambda_j \Delta z_j} D_{j+1}^- \\ D_j^- &= -J_{B,j}^T e^{i\omega\Lambda_j \Delta z_j} U_{j+1}^- + J_{A,j}^T e^{-i\omega\Lambda_j \Delta z_j} D_{j+1}^- \end{aligned} \tag{14}$$

Define reflection and transmission matrices Γ_j, T_j by

$$U_j^- = \Gamma_j D_j^-, U_j^+ = T_j D_j^- \tag{15}$$

From (14), (15) we obtain by induction

$$\begin{aligned} \Gamma_j &= \left(J_{A,j}^T \tilde{\Gamma}_{j+1} - J_{B,j}^T\right)\left(-J_{B,j}^T \tilde{\Gamma}_{j+1} + J_{A,j}^T\right)^{-1} \\ T_j &= T_{j+1} e^{i\omega\Lambda_j \Delta z_j}\left(-J_{B,j}^T \tilde{\Gamma}_{j+1} + J_{A,j}^T\right)^{-1} \end{aligned} \tag{16}$$

where $\tilde{\Gamma}_{j+1} = e^{i\omega\Lambda_j \Delta z_j}\Gamma_{j+1} e^{i\omega\Lambda_j \Delta z_j}$, and Γ_{j+1} is symmetric. Thus, all the reflection and transmission matrices can be calculated by (16), starting with (13).

4. Sources and boundary conditions. Consider a $2n$-dimensional system of the form (5) with $^{(m)}$ omitted. Let the source be of the form

$$S = S_0 \delta(z - z_s) + S_1 \delta'(z - z_s) \tag{17}$$

with S_0, S_1 independent of z. Define n-vectors

$$S_A, S_B : \left(S_A, S_B\right)^T = i\omega M S_1 - S_0$$

Using this formula we obtain the following jump condition across the source

$$\Phi(z_s^-) = \Phi(z_s^+) + \left(S_A, S_B\right)^T \tag{18}$$

Inserting a fictitious layer boundary at $z = z_s^+$ we compute the reflection matrix $\Gamma_s \equiv \Gamma(z_s^+)$. Since the material properties do not change at z_s, we have

$$\Psi(z_s^+) = \left(\Gamma_s D_s, D_s\right)^T \tag{19}$$

where $D_s \equiv D(z_s^+), U_s \equiv U(z_s^+)$. Using (9), (18) and (19) we obtain

$$\Psi(z_s^-) = \left(\Gamma_s D_s, D_s\right)^T + \frac{1}{\sqrt{2}}\left(L_2^T S_A + L_1^T S_B, L_2^T S_A - L_1^T S_B\right)^T$$

This expression may now be propagated upwards through layers, using (10) and jumped upwards across layers boundaries until we reach the free surface at $z = 0^+$. Then n boundary conditions at $z = 0$ can be used to find the n unknowns D_s.

Consider one particular case when $z_s \in (0, z_1)$. In this case

$$\begin{aligned} \Psi(0^+) = &\left(e^{i\omega\Lambda z_s}\Gamma_s D_s, e^{-i\omega\Lambda z_s} D_s\right)^T + \\ &+ \frac{1}{\sqrt{2}}\left(e^{i\omega\Lambda z_s}\left(L_2^T S_A + L_1^T S_B\right), e^{-i\omega\Lambda z_s}\left(L_2^T S_A - L_1^T S_B\right)\right)^T \end{aligned} \tag{20}$$

Define

$$\Phi(0^+) = \left(G_A \Phi_0, G_B \Phi_0\right)^T \tag{21}$$

For System 1, let

$$\Phi_0^{(1)} = \left(\tilde{v}_3, -\tilde{q}_3, \tilde{v}_1\right)_{z=0^+}^T$$

$$G_A^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, G_B^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We can check that (21) holds for System 1 with the boundary conditions $\tilde{\tau}_{13} = \tilde{\tau}_{33} = \tilde{p} = 0$ at the free surface $z = 0$.

For System 2, let

$$\Phi_0^{(2)} = \tilde{v}_2(0^+), \; G_A^{(2)} = 1, G_B^{(2)} = 0$$

Then it may be checked that (21) holds for System 2 with the boundary condition $\tilde{\tau}_{23} = 0$ at the free surface $z = 0$. Now using (9), (20) and (21) we obtain

$$\Phi_0 = \left(e^{i\omega\Lambda z_s}\Gamma_s e^{i\omega\Lambda z_s}\left(L_2^T G_A - L_1^T G_B\right) - \left(L_2^T G_A + L_1^T G_B\right)\right)^{-1} \times$$
$$\times e^{i\omega\Lambda z_s}\left(\Gamma_s\left(L_2^T S_A - L_1^T S_B\right) - \left(L_2^T S_A + L_1^T S_B\right)\right)$$
$$D_s = \frac{1}{\sqrt{2}} e^{i\omega\Lambda z_s}\left(L_2^T G_A - L_1^T G_B\right)\Phi_0 - \frac{1}{\sqrt{2}}\left(L_2^T S_A - L_1^T S_B\right) \quad (22)$$

In particular, when $z_s = 0^+$ we get

$$\Phi_0 = \left((\Gamma_s - I)L_2^T G_A - (\Gamma_s + I)L_1^T G_B\right)^{-1} \times$$
$$\times\left((\Gamma_s - I)L_2^T S_A - (\Gamma_s + I)L_1^T S_B\right) \quad (23)$$

Φ_0 defines all of Φ at the free surface, and $D_s, U_s = \Gamma_s D_s$ give all of Φ just below the source. Now we are able theoretically to compute Φ in any $z \in \mathbb{R}_+$ by propagating trough the layers using (10) and (11).

Remark. *Propagation of an upward-going wave in the downward direction will be unstable numerically using (10), because the complex exponentials grow rather than decay with distance. Then numerically, one has to obtain U from D using Γ_j, or the transmission matrix T_j.*

Once $\Phi^{(1)}$ and $\Phi^{(2)}$ have been determined, we may compute $\tilde{q}_1, \tilde{q}_2, \tilde{\tau}_{11}, \tilde{\tau}_{12}, \tilde{\tau}_{22}$ using (33), see Appendix.

Inverting the rotation transform, we can calculate the hat (^) variables, i.e.,

$$\hat{v} = \Omega^T \tilde{v}, \hat{q} = \Omega^T \tilde{q}, \hat{\tau} = \Omega^T \tilde{\tau}\Omega, \hat{p} = \tilde{p} \quad (24)$$

The matrices for Systems 1 and 2 depend only on the magnitude k. However, the transformation (24) depends on k_1, k_2. For any function $\hat{\xi}(k)$ let

$$\Xi_{j_1,j_2}\left(\hat{\xi}\right) \equiv F_{x_1x_2}^{-1}\left(k_1^{j_1}k_2^{j_2}\hat{\xi}(k)\right) = (-i)^{j_1+j_2}\,\partial_{x_1}^{j_1}\partial_{x_2}^{j_2}F_{x_1x_2}^{-1}\left(\hat{\xi}(k)\right)$$

We can compute these quantities as Hankel transforms in the cylindrical coordinates r, θ, z. Define

$$B_{j_1,j_2}(\hat{\xi}) = \frac{1}{2\pi}\int_0^\infty k^{j_1} J_{j_2}(kr)\hat{\xi}(k)dk$$

where J_n is the Bessel function and j_1, j_2 are nonnegative integers. Then

$$\Xi_{0,0} = B_{1,0}, \Xi_{1,0} = i\cos\theta B_{2,1}, \Xi_{0,1} = i\sin\theta B_{2,1}$$
$$\Xi_{1,1} = \sin\theta\cos\theta\left(B_{3,0} - \frac{2}{r}B_{2,1}\right)$$
$$\Xi_{2,0} = \cos^2\theta B_{3,0} - \frac{\cos 2\theta}{r}B_{2,1} \quad (25)$$
$$\Xi_{0,2} = \sin^2\theta B_{3,0} + \frac{\cos 2\theta}{r}B_{2,1}$$

These formulas are used to get the solution in real space.

Examples

1. Dynamite source. A dynamite source imposed on the solid and the fluid can be defined in the following form

$$f(x) = g(x) = -h(\omega)\nabla\delta(x - x_s)$$

where δ is the Dirac function, $x_s = (0,0,z_s)$ is the source position and $h(\omega)$ is the spectrum of the seismic moment. Applying the Fourier transform $F_{x_1x_2}$ we obtain

$$\hat{f} = \hat{g} = -h(\omega)\left(ik_1\delta(z - z_s), ik_2\delta(z - z_s), \delta'(z - z_s)\right)^T$$

and rotation by Ω yields

$$\tilde{f} = \tilde{g} = -h(\omega)\left(ik\delta(z - z_s), 0, \delta'(z - z_s)\right)^T \quad (26)$$

Substitution of (26) into the source expression yields the source for System 1, in the form of (17), with

$$S_0^{(1)} = h(\omega)\left(0, ik - \frac{\omega\rho_f k}{D - i\omega\rho_w}, \frac{k^2}{D - i\omega\rho_w}, 0,0,0\right)^T \quad (27)$$
$$S_1^{(1)} = h(\omega)\left(0,0,0,1,0,-1\right)^T$$

Substitution of (26) into the source expression for System 2 shows that $S^{(2)}$ is zero, and then $\dot{\tilde{u}}_2, \tilde{\tau}_{23}$ are zero too. This is to be expected result because System 2 is related to SH-waves, which are not excited by the dynamite source. Substitution of (27) into $\left(S_A, S_B\right)^T = i\omega M S_1 - S_0$ gives

$$S_A^{(1)} = i\beta h(\omega)\left(\omega(C - M), 2kG(M - C), \omega(\lambda_c + 2G - C)\right)^T$$
$$S_B^{(1)} = \left(0,0,0\right)^T \quad (28)$$

Formulas (28) may be used in (22) or (23) for a shallow source, to obtain all the tilde (~) functions. To invert rotation Ω, using (24), note that $\tilde{v}_2, \tilde{q}_2, \tilde{\tau}_{12}, \tilde{\tau}_{23}$ are identically zero. All the remaining tilde (~) functions depend of k only and can be calculated by the following formulas

$$\hat{v}_1 = \frac{k_1}{k}\tilde{v}_1, \hat{v}_2 = \frac{k_2}{k}\tilde{v}_1, \hat{v}_3 = \tilde{v}_3$$
$$\hat{q}_1 = \frac{k_1}{k}\tilde{q}_1, \hat{q}_2 = \frac{k_2}{k}\tilde{q}_2, \hat{q}_3 = \tilde{q}_3$$
$$\hat{\tau}_{11} = \frac{k_1^2\tilde{\tau}_{11} + k_2^2\tilde{\tau}_{22}}{k^2}, \hat{\tau}_{12} = \frac{k_1k_2(\tilde{\tau}_{11} - \tilde{\tau}_{22})}{k^2} \quad (29)$$
$$\hat{\tau}_{22} = \frac{k_2^2\tilde{\tau}_{11} + k_1^2\tilde{\tau}_{22}}{k^2}, \hat{\tau}_{13} = \frac{k_1\tilde{\tau}_{13}}{k}$$
$$\hat{\tau}_{23} = \frac{k_2\tilde{\tau}_{13}}{k}, \hat{\tau}_{33} = \tilde{\tau}_{33}, \hat{p} = \tilde{p}$$

Then the Fourier transforms $F_{x_1x_2}$ can be inverted in cylindrical coordinates (r,θ,z) using (25) to obtain the solid and fluid velocities

$$\begin{aligned} v &= \left(iB_{1,1}(\tilde{v}_1)\right)e_r + \left(B_{1,0}(\tilde{v}_3)\right)e_z \\ q &= \left(iB_{1,1}(\tilde{q}_1)\right)e_r + \left(B_{1,0}(\tilde{q}_3)\right)e_z \end{aligned} \qquad (30)$$

where e_r, e_z are unit vectors in the r,z coordinate directions, respectively, and the stress tensor components and the pressure

$$\begin{aligned} &\tau_{11} = \Xi_{2,0}(k^{-2}\tilde{\tau}_{11}) + \Xi_{0,2}(k^{-2}\tilde{\tau}_{22}), \tau_{12} = \Xi_{1,1}(k^{-2}(\tilde{\tau}_{11} - \tilde{\tau}_{22})) \\ &\tau_{22} = \Xi_{0,2}(k^{-2}\tilde{\tau}_{11}) + \Xi_{2,0}(k^{-2}\tilde{\tau}_{22}), \tau_{13} = \Xi_{1,0}(k^{-1}\tilde{\tau}_{13}) \\ &\tau_{23} = \Xi_{0,1}(k^{-1}\tilde{\tau}_{13}), \tau_{33} = \Xi_{0,0}(\tilde{\tau}_{33}), p = \Xi_{0,0}(\tilde{p}) \end{aligned} \qquad (31)$$

2. Vertical source. We next consider a vertical point source acting on the free surface $z = 0$, i.e.,

$$f(x) = g(x) = (0,0,1)^T h(\omega)\delta(x_1)\delta(x_2)\delta(z - z_s)$$

where $z_s \to 0^+$ puts the force on the free surface. This models hammer, weight drop, and vibroseis sources.

Applying the Fourier transforms $F_{x_1x_2}$ and rotation Ω we arrive at

$$\tilde{f} = \tilde{g} = \hat{f} = \hat{g} = (0,0,1)^T h(\omega)\delta(z - z_s) \qquad (32)$$

Substitution of (32) into the source expressions for Systems 1 and 2 yields

$$S^{(1)} = (0,0,0,-1,0,1)^T h(\omega)\delta(z - z_s), S^{(2)} = (0,0)^T$$

Thus, all the variables in System 2 are zero, as it was in the case of dynamite source. From (17) and definition of S_A, S_B we obtain

$$S_A^{(1)} = \left(0,0,0\right)^T, S_B^{(1)} = \left(1,0,-1\right)^T h(\omega)$$

Now all the tilde variables at the free surface may be computed using (23) as $z_s \to 0^+$ and propagated anywhere else in space. Note $S_A^{(1)}, S_B^{(1)}$ are independent of k_1, k_2, so the tilde variables depend only on k and not on wave number direction. Therefore, similar to dynamite we can transform to the hat variables using (29) and transform back to the spatial variables using (30)-(31).

Conclusions

Based on the Ursin method, we have shown how the complete Biot-JKD equations (higher-frequency range) can be put into the Ursin form in a plane-layered medium. We have derived explicit formulas of the solution to a boundary-value problem formulated for these equations.

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Appendix

System 1.
$$M_1^{(1)} = \begin{pmatrix} -\beta M & \beta\gamma(C^2-\lambda_c M) & -\beta C \\ \beta\gamma(C^2-\lambda_c M) & \rho+\dfrac{i\omega\rho_f^2}{D-i\omega\rho_w}-4\beta\gamma^2 G(C^2-M(\lambda_c+G)) & 2\beta\gamma GC-\dfrac{i\omega\rho_f\gamma}{D-i\omega\rho_w} \\ -\beta C & 2\beta\gamma GC-\dfrac{i\omega\rho_f\gamma}{D-i\omega\rho_w} & -\beta(\lambda_c+2G)+\dfrac{i\omega\gamma^2}{D-i\omega\rho_w} \end{pmatrix}$$

$$M_2^{(1)} = \begin{pmatrix} \rho & \gamma & -\rho_f \\ \gamma & G^{-1} & 0 \\ -\rho_f & 0 & -\dfrac{D-i\omega\rho_w}{i\omega} \end{pmatrix}, S^{(1)} = (0,-\tilde{f}_1-\frac{i\omega\rho_f}{D-i\omega\rho_w}\tilde{g}_1,\frac{ik}{D-i\omega\rho_w}\tilde{g}_1,-\tilde{f}_3,0,\tilde{g}_3)^T$$

There are three modes: fast compressional wave ($\lambda_1^{(1)}$), Biot slow wave ($\lambda_2^{(1)}$), and vertical shear wave ($\lambda_3^{(1)}$).

Eigenvalues:

$$(\lambda_j^{(1)})^2 = -\gamma^2+\beta\left(C\rho_f-\frac{M\rho}{2}+(\lambda_c+2G)\frac{D-i\omega\rho_w}{2i\omega}\right)\pm\frac{\beta}{2}\sqrt{\left((\lambda_c+2G)\frac{D-i\omega\rho_w}{i\omega}+M\rho\right)^2-4\left(M\rho_f+C\frac{D-i\omega\rho_w}{i\omega}\right)\left(C\rho-(\lambda_c+2G)\rho_f\right)}, j=1,2$$

$$\text{(with (+) for m=1 and (-) for m=2), and } (\lambda_3^{(1)})^2 = -\gamma^2+G^{-1}\left(\rho+\frac{i\omega\rho_f^2}{D-i\omega\rho_w}\right)$$

Eigenvectors:

$$a_j^{(1)} = \bar{a}_j\left(-1,2G\gamma,\zeta_j\right)^T, j=1,2, a_3^{(1)} = \frac{\bar{a}_3}{\lambda_3^{(1)}}\left(\gamma, G(\lambda_3^{(1)})^2-G\gamma^2,\frac{i\omega\gamma\rho_f}{D-i\omega\rho_w}\right)^T$$

$$b_j^{(1)} = \frac{\bar{a}_j}{\lambda_j^{(1)}}\left(2G\gamma^2-\rho-\rho_f\zeta_j,\gamma,\rho_f-\zeta_j\frac{D-i\omega\rho_w}{i\omega}\right)^T, j=1,2, b_3^{(1)} = \bar{a}_3\left(2G\gamma,1,0\right)^T$$

where

$$\zeta_j = \frac{C\rho-(\lambda_c+2G)\rho_f}{\dfrac{(\lambda_j^{(1)})^2+\gamma^2}{\beta}-C\rho_f-(\lambda_c+2G)\dfrac{D-i\omega\rho_w}{i\omega}}, \bar{a}_j = \sqrt{\frac{\lambda_j^{(1)}}{\rho+2\rho_f\zeta_j-\zeta_j^2\dfrac{D-i\omega\rho_w}{i\omega}}}, j=1,2, \bar{a}_3 = \sqrt{\frac{\lambda_3^{(1)}}{G\gamma^2+G(\lambda_3^{(1)})^2}}$$

System 2. $M_1^{(2)} = G^{-1}, M_2^{(2)} = \rho-G\gamma^2+\dfrac{i\omega\rho_f^2}{D-i\omega\rho_w}, S^{(2)} = (0,-\tilde{f}_2-\dfrac{i\omega\rho_f}{D-i\omega\rho_w}\tilde{g}_2)^T$. There is the horizontal shear wave mode only ($\lambda^{(2)}$) with

$$(\lambda^{(2)})^2 = -\gamma^2+G^{-1}\left(\rho+\frac{i\omega\rho_f^2}{D-i\omega\rho_w}\right), a^{(2)} = \sqrt{\frac{1}{G\lambda^{(2)}}}, b^{(2)} = \sqrt{G\lambda^{(2)}}$$

Here $\beta = (C^2-M(\lambda_c+2G))^{-1}$. Dependent variables are calculated by the following formulas:

$$\tilde{q}_1 = \frac{1}{D-i\omega\rho_w}(-ik\tilde{p}+i\omega\rho_f\tilde{v}_1+\tilde{g}_1), \tilde{q}_2 = \frac{i\omega\rho_f}{D-i\omega\rho_w}\tilde{v}_2+\frac{1}{D-i\omega\rho_w}\tilde{g}_2, \tilde{\tau}_{12} = -G\gamma\tilde{v}_2$$
$$\tilde{\tau}_{11} = \beta\left\{-4\gamma G(C^2-M(\lambda_c+G))\tilde{v}_1+(C^2-\lambda_c M)\tilde{\tau}_{33}+2GC\tilde{p}\right\}, \tilde{\tau}_{22} = \beta\left\{-2\gamma G(C^2-\lambda_c M)\tilde{v}_1+(C^2-\lambda_c M)\tilde{\tau}_{33}+2GC\tilde{p}\right\} \quad (33)$$