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A time-fractional viscoacoustic wave equation based on complex phase velocity expression

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Abstract Summary

Accurate wave propagation modeling is essential for high-resolution seismic imaging. Traditional time-domain models typically employ integer-order derivatives. In this study, we propose a time-domain fractional-order wave equation based on a fractional polynomial approximation of velocity. This formulation enhances model accuracy over a broader frequency range and for lower quality factor (Q) values. Numerical experiments demonstrate that the proposed model shows good agreement with analytical solutions.

Introduction

Accurate modeling of viscoacoustic wave propagation is essential for high-resolution seismic imaging and other geophysical applications. In this context, many studies have sought equations that better describe wave propagation in attenuating media (Yang et al., 2014). Recently, Yang and Zhu (2018) derived a viscoacoustic wave equation using Aki & Richards' (1980) phase velocity expression. This formulation decouples dispersion and dissipation, simplifying independent analysis of these effects. However, their second-order polynomial approximation of the logarithmic dispersion term introduces significant errors at low frequencies and low Q-values. In this work, we propose a time-fractional viscoacoustic wave equation using Aki & Richards' phase velocity expression by modifying the logarithmic dispersion term by using a generalized logarithmic function from Sharma-Mittal-Taneja entropy theory (Mittal, 1975; Sharma and Taneja, 1975). This generalized logarithm can better approximate dispersion effects, enabling accurate viscoacoustic modeling across all frequencies and Q-values. It is important to note that solving equations in the time domain offers several key advantages. Time-domain methods are particularly well-suited for analyzing nonlinear systems, which can be difficult to handle in the frequency domain. Furthermore, time-domain approaches are widely used in industrial practice.

Theory

Following Aki and Richards (1980), wave propagation in attenuating media can be represented by a complex velocity field that varies with frequency, as follows:

$$v(\mathbf{x},\omega) = v_0(\mathbf{x})\left[1 + \frac{1}{\pi Q(\mathbf{x})}\ln\left(\frac{\omega}{\omega_0}\right) - \frac{i}{2\,Q(\mathbf{x})}\right] \tag{1}$$

where $v_0(\mathbf{x})$ is the velocity at the reference frequency ω_0, Q denotes the quality factor, and $\mathbf{x}$ represents the spatial coordinates. The real part models phase dispersion, while the imaginary term quantifies energy loss (dissipation effects).

Substituting Eq. (1) into the wave equation,

$$-\frac{\omega^2}{\rho(\mathbf{x})v^2(\mathbf{x},\omega)}P_s(\mathbf{x},\omega) - \nabla\cdot\left[\frac{1}{\rho(\mathbf{x})}\nabla P_s(\mathbf{x},\omega)\right] = F_s(t)\delta(\mathbf{x}-\mathbf{x}_s), \tag{2}$$

and neglecting the second-order terms in Q, generates an expression that is not straightforward to transform into the time domain due to the presence of a second-order logarithmic term $\omega^2 \ln\left(\frac{\omega}{\omega_0}\right)$. To address this, we employ the geometric series approximation $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \approx 1+r$, as proposed by Yang and Zhu (2018).After applying the geometric series expansion, we obtain the expression on the left-hand side of Equation 3. Yang and Zhu (2018) introduces an additional approximation by employing a second-order polynomial, as shown in Equation 3.

$$w^2 ln\left(\frac{w}{w_0}\right) \approx aw^2 + bw + c \tag{3}$$

To better capture low-frequency behavior and achieve an improved fit for systems with a low Q-factor, we replace the second-order polynomial with a fractional-order polynomial as given in Equation 4.

$$w^2 ln\left(\frac{w}{w_0}\right) \approx \frac{c}{a-b}w^{2-b} - \frac{c}{a-b}w^{2-d-a} \tag{4}$$

Fractional polynomials for function approximation are widely found in the literature (Euler, 1783), (Mittal, 1975), (Sharma and Taneja, 1975), (da Silva and Kaniadakis, 2021), and (Janine, 2025). In Equation 4, the parameter $\langle d \rangle$ was introduced to optimize the curve fitting.
Finally, by substituting expression 4 into the frequency-domain wave equation 2 and applying the geometric series approximation, we can perform the inverse Fourier transform to obtain the fractional-order time-domain wave equation 5.

$$-\frac{1}{\rho v_0^2}\left[d_0\frac{\partial^2}{\partial t^2} + d_1\frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}} + d_2\frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}}\right]p(x,t) - \nabla\cdot\left[\frac{1}{\rho}\nabla p(x,t)\right] = f(x_s,t)\delta(x-x_s) \tag{5}$$

Where

$$\alpha_1 = 2-b, \;\; \alpha_2 = 2-d-a, \;\; d_0 = -\frac{i+Q}{Q}, \;\; d_1 = -\frac{c}{a-b}\frac{2\,i^{-\alpha_1}}{\pi\,Q} \;\; and \;\; d_2 = \frac{c}{a-b}\frac{2\,i^{-\alpha_2}}{\pi\,Q} \tag{6}$$

Results

The parameters of the fractional polynomial in Equation 4, obtained through curve fitting and subsequently substituted into Equation 6, are summarized as follows:

$$\alpha_1 = 2-b = 2-0.4112 = 1.5888, \;\; \alpha_2 = 2-d-a = 2-0.8671-(-1.063) = 2.1959,$$

$$d_0 = -\frac{i+Q}{Q}, \;\; d_1 = \frac{12098\,i^{-1.5888}}{4559\,\pi\,Q} \;\; and \;\; d_2 = -\frac{12098\,i^{-2,1959}}{4559\,\pi\,Q} \tag{7}$$

After establishing the time-domain fractional-order wave equation, we proceeded to solve it in a homogeneous medium. The literature presents several definitions of fractional derivatives, including those of Riemann–Liouville, Caputo, and Grünwald–Letnikov (Camargo, 2009). In this study, we adopt the Grünwald–Letnikov definition due to its suitability for implementation via finite difference methods (FDM).
The input data used in the simulations are summarized as follows: $w_0 = 2\pi\, rad/s$, $\rho = 2000\, kg/m^3$, $v_0 = 3000\; m/s$, $0 \le x \le 3000\; m$, $0 \le z \le 3000\; m$, $\Delta x = \Delta z = 7.5\; m$ and source

location $(x_s = z_s = 1500\ m)$. In the simulations, variations were applied to the following parameters: $\Delta t = 1\ ms$, $\Delta t = 0.2\ ms$, $Q = 20$, $Q = 80$, receptor number 1 location $(x_{r_1} = 2175\ m,\ z_{r_1} = 1500\ m)$, receptor number 2 location $(x_{r_2} = 2325\ m,\ z_{r_2} = 1500\ m)$, and the number of terms used in the fractional derivative approximation $(20 \leq n \leq 150)$.
The following plots illustrate the analytical solution of the wave equation, the numerical solution incorporating a fractional derivative, and the numerical solution employing an integer-order derivative. We employed the Pearson correlation coefficient and the mean squared error as quantitative metrics to compare the analytical solution with the fractional derivative solution.

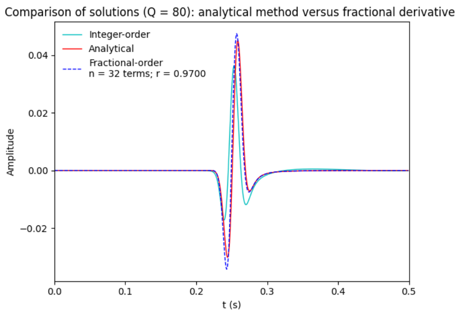

Figure 1: Wave equation solution for $x_{r_1} = 2175\ m,\ \Delta t = 1\ ms$ - Pearson metric.

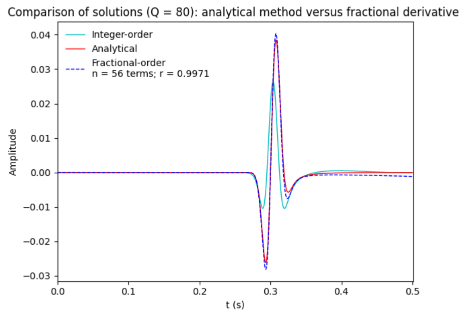

Figure 2: Wave equation solution for $x_{r_2} = 2325\ m,\ \Delta t = 0.2\ ms$ - Pearson metric.

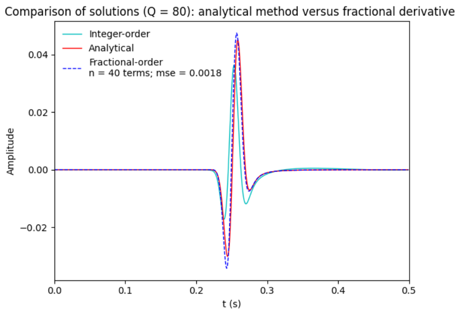

Figure 3: Wave equation solution for $x_{r_1} = 2175\ m,\ \Delta t = 1\ ms$ - MSE metric.

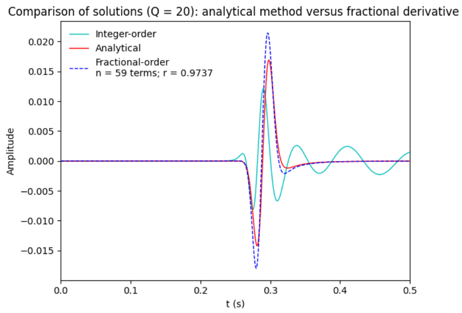

Figure 4: Wave equation solution for $x_{r_2} = 2325\ m,\ \Delta t = 1\ ms$ - Pearson metric.

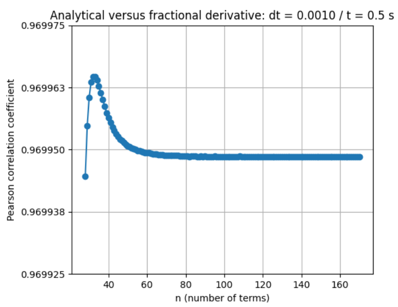

Figure 5: Pearson correlation coefficient as a function of the number of terms in the fractional derivative: $Q = 80,\ x_{r_1} = 2175\ m,\ \Delta t = 1\ ms$.

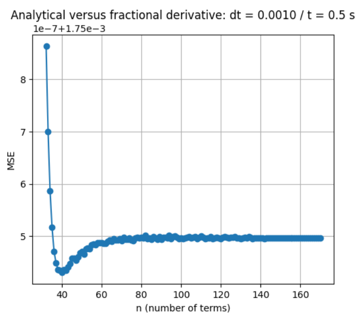

Figure 6: Mean squared error as a function of the number of terms in the fractional derivative: $Q = 80,\ x_{r_1} = 2175\ m,\ \Delta t = 1\ ms$.

Conclusions

In the simulations performed, we observed that the model incorporating the fractional derivative more closely approximates the analytical solution compared to the integer-order derivative model, when the time step (Δt) and the number of terms in the fractional derivative (n) are properly selected. The main drawback of the fractional model is its higher computational cost, which can be addressed through parallel processing, a subject for future work. Another key area for future research is the investigation of the numerical stability of the fractional model, considering the parameters time step (Δt), quality factor (Q), and the number of terms in the fractional derivative (n).

Acknowledgments

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