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Viscoacoustic Wave Modeling with a Time-Domain Complex-Valued Equation Using Tsallis q-Logarithm

Sofia Severo Galvão (LAPPS; UFRN; Brazil), Sergio Da Silva ([1]LAPPS; UFRN; Brazil; [2]CNR; PoliTO; Italy), Carlos Da Costa (LAPPS; UFRN; Brazil), Tiago Barros (Universidade Federal do Rio Grande do Norte), Samuel Xavier-de-Souza (LAPPS; UFRN; Brazil)

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Abstract Summary

This work introduces a novel time-domain complex-valued viscoacoustic wave equation based on the Tsallis *q*-logarithm function to model dissipation and dispersion effects in seismic wave propagation. The *q*-logarithm, a generalization of the standard logarithmic function, allows for the exploration of alternative physical models of seismic wave dispersion, in which the phase velocity depends on both the quality factor Q and the hyperparameter q, as defined within Tsallis nonextensive statistical mechanics. We compare the proposed *q*-logarithmic-based model with the recently introduced second-order approximation of the standard logarithmic function. Our results show that the proposed model provides a more accurate representation of dissipation and dispersion, particularly in highly attenuating media (i.e., with very low Q-factors), where it outperforms traditional quadratic approximations when benchmarked against the analytical Green's function in a homogeneous medium.

Introduction

The heterogeneous and anelastic subsurface attenuates seismic waves by scattering energy and converting it into heat. This attenuation process, usually quantified by the quality factor Q, not only reduces seismic amplitude but also induces velocity dispersion, distorting waveforms. Over the past decades, researchers have extensively studied dissipation and dispersion (Yang et al., 2014), proposing various viscoacoustic models ranging from classical constant-Q formulations (Futterman, 1962; Kjartansson, 1979) to modern decoupled approaches that explicitly separate amplitude attenuation and phase dispersion (Mu et al., 2025; Yang and Zhu, 2018).

In this work, we propose a novel viscoacoustic wave equation based on Tsallis *q*-statistics (Tsallis, 1988), offering a flexible framework for modeling attenuation and dispersion. Originally developed in statistical physics, Tsallis *q*-statistics have successfully captured complex phenomena (Tsallis, 2023), including applications in seismic inversion (Bassrei and Quezada, 2001; da Silva et al., 2022). Specifically, we derive a time-domain complex-valued viscoacoustic wave equation for frequency-independent Q media by generalizing the dispersion term of the frequency-dependent complex velocity introduced by Aki and Richards (2002) using the Tsallis *q*logarithm function, initially incorporating it into the frequency-domain formulation. By strategically tuning the hyperparameter q, we obtain a time-domain formulation that avoids the need for computationally expensive fractional derivatives and additional mathematical approximations.

Theory

Considering first-order approximation in the quality factor Q, we propose that the frequency-dependent complex velocity can be expressed as:

$$v(\mathbf{x},\omega) = v_0(\mathbf{x})\left[1 + \frac{1}{\pi Q(\mathbf{x})}\ln_q\left(\frac{\omega}{\omega_0}\right) - \frac{i}{2Q(\mathbf{x})}\right], \quad \text{with} \quad \ln_q(x) = \frac{x^{1-q}-1}{1-q}, \tag{1}$$

where $v(\mathbf{x},\omega)$ denotes the P-wave velocity, which varies with the spatial position $\mathbf{x}$ and the angular frequency $\omega > 0$. The reference velocity v_0 corresponds to the reference angular frequency ω_0, and $\ln_q(x)$ represents *q*-logarithm (Tsallis, 1988), for which the classical logarithm emerges as a special case in the limit as $q \to 1$. In this classical limit, the velocity defined in Eq. (1) reduces to the form proposed by Aki and Richards (2002): $v(\mathbf{x},\omega) = v_0(\mathbf{x})\left[1 + \frac{1}{\pi Q(\mathbf{x})}\ln\left(\frac{\omega}{\omega_0}\right) - \frac{i}{2Q(\mathbf{x})}\right]$.

Inserting Eq. (1) into the frequency-domain acoustic wave equation and considering only the first-order terms in Q yields

$$\nabla\cdot\left(\frac{1}{\rho(\mathbf{x})}\nabla P(\mathbf{x},\omega)\right) + \frac{\omega^2}{\rho(\mathbf{x})v_0^2(\mathbf{x})}\left\{1 - \frac{2\left[\left(\frac{\omega}{\omega_0}\right)^{1-q} - 1\right]}{(1-q)\pi Q(\mathbf{x})} + \frac{i}{Q(\mathbf{x})}\right\}P(\mathbf{x},\omega) = -F(\mathbf{x},\omega), \quad (2)$$

where $P(\mathbf{x},\omega)$ is the frequency-domain pressure wavefield, $\rho(\mathbf{x})$ is the mass density, and $F(\mathbf{x},\omega)$ is the source function in frequency domain.

Following the approximation $\omega \approx |\mathbf{k}|c$, we approximate the dissipation term in Eq. (2) as follows (Zhu and Harris, 2014):

$$\frac{i\omega^2}{\rho v_0^2 Q} \approx \frac{i\omega|k|}{\rho v_0 Q} \xrightarrow{\mathcal{F}_{\mathbf{x}}} \frac{i\omega\sqrt{-\nabla^2}}{\rho v_0 Q}, \quad (3)$$

where $|\mathbf{k}|$ is the magnitude of the complex-valued wavenumber, $\sqrt{-\nabla^2}$ represents a pseudo-differential operator, and $\mathcal{F}_{\mathbf{x}}$ denotes the forward spatial Fourier transform.

Substituting Eq. (3) into Eq. (2) and applying the inverse Fourier transform, we derive a viscoacoustic wave equation in the space-time domain as a function of the parameter *q*:

$$\frac{1}{\rho(\mathbf{x})v_0^2(\mathbf{x})}\left[\left(1 + \frac{2}{\pi Q(\mathbf{x})(1-q)}\right)\frac{\partial^2}{\partial t^2} - \left(\frac{2\tau^{1-q}}{\pi Q(\mathbf{x})(1-q)}\right)i^{(q-1)}\frac{\partial^{(3-q)}}{\partial t^{(3-q)}}\right]p(\mathbf{x},t)$$
$$-\left[\frac{1}{\rho(\mathbf{x})v_0(\mathbf{x})Q(\mathbf{x})}\frac{\partial}{\partial t}\sqrt{-\nabla^2}\right]p(\mathbf{x},t) - \nabla\cdot\left[\frac{1}{\rho(\mathbf{x})}\nabla p(\mathbf{x},t)\right] = f(\mathbf{x_s},t). \quad (4)$$

where $p(\mathbf{x},t)$ and $f(\mathbf{x}_s,t)$ denote the time-domain pressure wavefield and the source term, respectively, with $\mathbf{x}_s$ representing the position of the seismic source. In addition, we define $\tau = \omega_0^{-1}$.

In this work, we focus on the particular case $q = 2$, which significantly simplifies the mathematical formulation by eliminating the need for fractional time derivatives. This choice leads to the following viscoacoustic wave equation:

$$\frac{1}{\rho(\mathbf{x})v_0^2(\mathbf{x})}\left[\left(1 - \frac{2}{\pi Q(\mathbf{x})}\right)\frac{\partial^2}{\partial t^2} + \left(\frac{2}{\pi Q(\mathbf{x})\tau}\right)i\frac{\partial}{\partial t}\right]p(\mathbf{x},t)$$
$$-\left[\frac{1}{\rho(\mathbf{x})v_0(\mathbf{x})Q(\mathbf{x})}\frac{\partial}{\partial t}\sqrt{-\nabla^2}\right]p(\mathbf{x},t) - \nabla\cdot\left[\frac{1}{\rho(\mathbf{x})}\nabla p(\mathbf{x},t)\right] = f(\mathbf{x_s},t). \quad (5)$$

Results

To test the performance of the proposed viscoacoustic model, we consider a two-dimensional (2D) homogeneous medium with a reference velocity of $3000\,m/s$, a density of $2.0\,g/cm^3$, and a reference frequency of 1 Hz. We use a Ricker wavelet with a peak frequency of 20 Hz as the seismic source. We placed the receiver 375 m from the central source location. For the numerical implementation, we apply an explicit eighth-order finite-difference scheme for the spatial derivatives and an explicit second-order scheme for the time derivatives on a regular Cartesian grid. In addition, we solve the pseudo-differential operator using the pseudospectral method. We compare our proposed model with $q = 2$ to the classical case studied in (Yang and Zhu, 2018), which uses the classical logarithm and approximates the term $\omega^2 \ln(\omega/\omega_0)$ with a second-degree polynomial.

In Figs. 1(a)–(c), we compare the analytical solution of the classical viscoacoustic wave equation to the numerical solution for quality factors $Q = 100$, $Q = 50$, and $Q = 20$. Here, we obtained the numerical results using Yang & Zhu's (2018) second-order polynomial approximation of the standard logarithm. The analytical and numerical curves diverge in highly attenuative media (e.g., $Q = 20$) due to this approximation's limitations at low frequencies. Panels (d)–(f) show the results from the proposed visco-acoustic model for the same Q-values. The q-logarithm-based model improves agreement between the analytical and numerical solutions, particularly in modeling dispersion across broader frequency ranges.

Figure 2 presents wavefield snapshots at $0.48\,s$ generated under identical model conditions as Fig. 1, but considering only the proposed wave equation. Panel (a) compares four distinct wave propagation regimes: (i) Acoustic ($Q \to \infty$), (ii) Dispersion-dominated, (iii) Dissipation-dominated, and

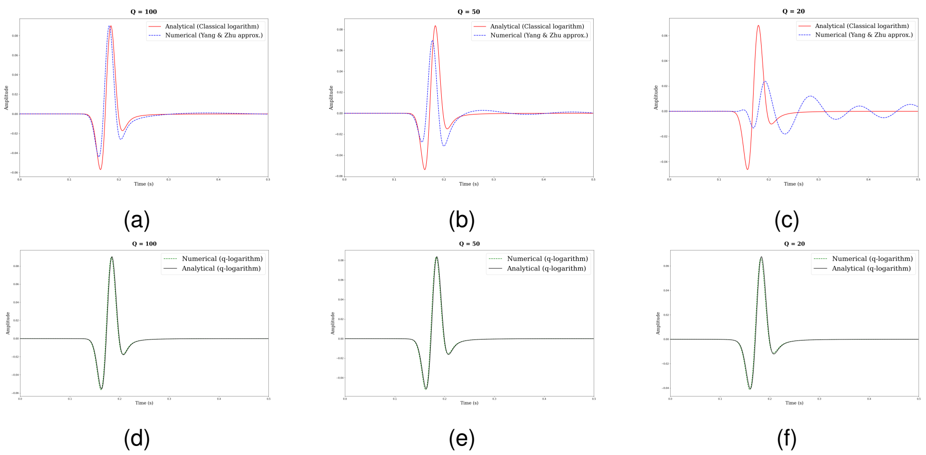

Figure 1: Comparison of numerical and analytical visco-acoustic solutions for modeling dispersion effects using (a–c) the standard logarithmic function and (d–f) the q-logarithmic function case for $q = 2$. Panels (a), (b), and (c) correspond to quality factors $Q = 100$, 50, and 20, respectively, using the classical case with polynomial approximation. Panels (d), (e), and (f) repeat the same Q-values but employ the proposed viscoacoustic model.

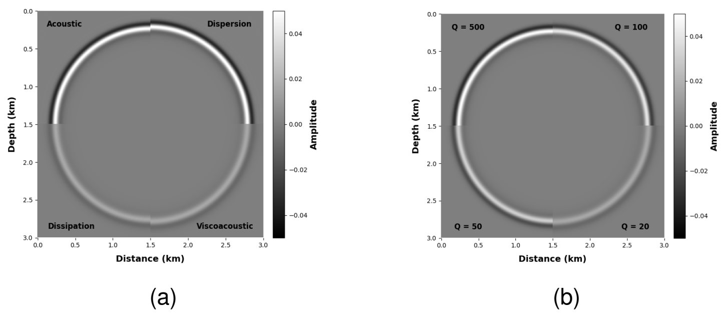

Figure 2: Comparison of wavefield snapshots at $0.48\,s$: (a) Acoustic, dispersion-dominated, dissipation-dominated, and visco-acoustic solutions for $Q = 20$, showing phase and amplitude effects; (b) Visco-acoustic wavefields for different Q values (100, 50, 20), demonstrating increasing attenuation. All simulations use a central point source with a 20 Hz Ricker wavelet.

(iv) Visco-acoustic (Combines both dispersion and dissipation effects). The dispersion-dominated solution for $Q = 20$ reveals, as expected, a noticeable phase lead compared to the acoustic case, while the dissipation-dominated wavefield displays significant amplitude reduction. The visco-acoustic quadrant synthesizes these behaviors, demonstrating both phase shift and amplitude attenuation. Panel (b) focuses exclusively on visco-acoustic solutions, contrasting wavefield characteristics across different quality factors ($Q = 100,\ 50,\ 20$). The snapshots clearly illustrate how decreasing Q values enhance attenuation effects, with $Q = 20$ showing the most pronounced amplitude decay and velocity dispersion.

Conclusions

In this paper, we have introduced a novel time-domain viscoacoustic wave equation based on Tsallis q-statistcs by replacing the classical logarithmic dispersion term by the q-logarithm. In particular, we have studied the case $q = 2$. The numerical experiments have shown that the proposed model outperforms the traditional polynomial approximations of the classical approach compared to its analytical solutions, especially in strongly damping media (low Q). The q-logarithmic formulation agreed better with the analytical solutions and minimized the phase and amplitude errors while maintaining the computational efficiency. In perspective, we intend to investigate the new physics arising from the different q values for different attenuation regimes and how this proposed model can improve wave equation-based modeling in geophysical applications.

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