Submission code: Z6XJ8P4965

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Iterative Solution of Helmholtz Equation Using Convergent Born Series

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Abstract

The scalar Helmholtz equation, governing wave propagation in the frequency domain for media with variable velocity and constant density, can be reformulated as a Lippmann-Schwinger integral equation. The resulting discretized linear system is large and computationally expensive to solve directly, making iterative methods more practical. The Traditional Born Series (TBS) offers a low-cost approximation but converges only for weak scattering potentials. To overcome this, modified scattering series with enhanced convergence, such as the Convergent Born Series (CBS), use preconditioners based on the scattering potential or generalized over-relaxation. This work introduces a Modified Born Series (MBS) within a reconditioned Richardson iteration, equivalent to CBS and convergent in strongly scattering media. Numerical tests compare MBS seismograms with time-domain extrapolation results, showing MBS as an effective alternative.

Introduction

The Helmholtz equation, governing scalar wave propagation in the frequency domain, is widely used to model seismic waves in heterogeneous media. Reformulated as a Lippmann–Schwinger integral equation, its discretization yields a large, computationally expensive linear system, making iterative methods more practical. The Traditional Born Series (TBS) (Morse and Feshbach, 1953) converges only for weak scattering potentials. To overcome this, Convergent Born Series (CBS) methods have been developed. Kleinman et al. (1988, 1990) proposed a Generalized Successive Over-Relaxation (GSOR) approach, while Osnabrugge et al. (2016) introduced a CBS using a preconditioner derived from the scattering potential. Xu et al. (2024) further improved convergence by combining GSOR with preconditioning. CBS ensures convergence in strongly scattering media by localizing the Green's function with damping and using a preconditioner. By adopting a homogeneous velocity background, the Laplacian operator is diagonalized in the wavenumber domain, reducing computational cost and avoiding discretization of derivatives, thus yielding highly accurate solutions. These features make CBS an effective tool for frequency-domain seismic inverse problems.

The CBS method uses a damped Green's function and splits the Helmholtz operator into background and perturbation terms. With a constant background velocity, the Laplacian becomes diagonal in the wavenumber domain, enabling fast, accurate iterations without discretizing derivatives. This efficiency makes CBS ideal for frequency-domain wave simulations and seismic inverse problems. In this work, we revisit CBS using the simplified formalism of Vettenburg and Vellekoop (2023) and propose a Modified Born Series (MBS) based on the same operator splitting. Incorporated into a preconditioned Richardson iteration, MBS retains CBS's convergence while offering an alternative formulation. Unlike finite-difference methods, MBS avoids numerical dispersion and discretization errors, ensuring high-fidelity seismic modeling. Numerical experiments compare MBS seismograms with those from the Rapid Expansion Method (REM), validating its effectiveness.

Theory

Traditional convergent Born series

The inhomogeneous Helmholtz equation can be numerically solved using a Born series, as demonstrated by Osnabrugge et al. (2016). They express the equation as

$$\left[\nabla^2 + \frac{\omega^2}{c_0^2} + i\epsilon\right] U(\mathbf{x},\omega) = -V(\mathbf{x},\omega)U(\mathbf{x},\omega) - S(\mathbf{x},\omega), \tag{1}$$

where $U(\mathbf{x},\omega)$ is the Fourier-transformed pressure wavefield, ∇^2 is the Laplacian operator, $c_0(\mathbf{x})$ is the background velocity, $S(\mathbf{x},\omega)$ is the source term, $V(\mathbf{x},\omega) = \omega^2 \left[\frac{1}{c^2(\mathbf{x})} - \frac{1}{c_0^2}\right] - i\epsilon$, is the scattering potencial, and ϵ is a small damping term. The Green's operator (G_0) in Fourier transformed coordinates $\mathbf{k}$, associated with the differential operator $L_0 = \left[|\mathbf{k}|^2 + \frac{\omega^2}{c_0^2} + i\epsilon\right]$ is given by

$$G_0(\mathbf{k},\omega) = \frac{1}{|\mathbf{k}|^2 - \omega^2/c_0^2 - i\epsilon}, \tag{2}$$

where $|\mathbf{k}|^2 = k_x^2 + k_y^2 + k_z^2$, and $G_0(\mathbf{x},\omega)$ satisfies $L_0 G_0 = -\delta(\mathbf{x})$, such that $G_0 = -L_0^{-1}$, with δ denoting the Dirac delta function.

Applying the Green's operator to both sides of Eq. 1 and considering a discrite grid yields $U = G_0VU + G_0S$. Repetead substitutuion of the right-hand side into U leads to the TBS solution: $U = [1 + G_0V + G_0VG_0V + \ldots]G_0S = \sum_{n=0}^{\infty}(G_0V)^n G_0S$. However, this series converges only if the magnitudes of all eigenvalues of G_0V are less than one. To address this limitation, Osnabrugge et al. (2016) proposed the preconditioner $\gamma(\mathbf{x},\omega) = iV(\mathbf{x},\omega)/\epsilon$, leading to the CBS solution: $U = [1 + M + M^2 + \cdots]\gamma G_0 S$ with $M = \gamma G_0 V + I - \gamma$, where I is the identity matrix. From this series, the iterative solution is derived as

$$U_{n+1} = MU_n + \gamma G_0 S, \quad \text{with} \quad U_0 = \gamma G_0 S. \tag{3}$$

The CBS is guaranteed to converge for scattering media with arbitrarily strong velocity constrats, provided that ϵ is appropriate chosen. Osnabrugge et al. (2016) showed that the series converges for all media if $\epsilon \geq \max\left|\frac{\omega^2}{c^2(\mathbf{x})} - \frac{\omega^2}{c_0^2}\right|$. The CBS algorithm can be implemented iteratively using the Fourier transform, with all multiplications performed element-wise. For each frequency, the quantities $V(\mathbf{x},\omega)$, $S(\mathbf{x},\omega;\mathbf{x}_s)$, ϵ and γ must be computed. The Green's function is also evaluted at each grid point in the frequency domain.

Modified Born series - Preconditioned Richardson iteration

Considering the inhomogeneous Helmhotz, for convinence we introduce the notation $A\mathbf{x} = \mathbf{y}$, where $A = \beta\left[\nabla^2 + \frac{\omega^2}{c^2(\mathbf{x})}\right]$, $\mathbf{y} = -\beta S(\mathbf{x},\omega)$ and β is a scalar factor. In both the TBS and CBS formalisms, a key step is to split the operator as $A = L_0 + V$, with $L_0 = \beta\left[\nabla^2 + \frac{\omega^2}{c_0^2}\right]$ representing the wave operator for a homogeneous medium with constant velocity c_0, and $V = \beta\left[\frac{\omega^2}{c^2(\mathbf{x})} - \frac{\omega^2}{c_0^2}\right]$ being the scattering potencial. Now, based on this spliting, we solve the Helmhotz equation using a preconditioned Richardson iteration

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \alpha\Gamma^{-1}(\mathbf{y} - A\mathbf{x}_n), \tag{4}$$

where $\Gamma = (L_0+I)(I-V)^{-1}$ is the preconditioner. The constant α is called the relaxation parameter. The proposed MBS formulation is equivalent to the CBS method introduced by Osnabrugge et al. (2016). For further details, the reader is referred to Swapnil and Vellekoop (2024). Substituting $\Gamma^{-1} = (I - V)(L_0 + I)$ and $A = (L_0 + I) - (I - V)$ into Eq. 4 yields the iterative solution

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \alpha(I - V)\left(\mathbf{x}_n - (L_0 + I)^{-1}[\mathbf{y} + (I - V)\mathbf{x}_n]\right). \tag{5}$$

The operator $(L_0 + I)$ is spatially invariant and can therefore be efficiently inverted using

$$(L_0 + I)^{-1} = \text{IFFT}\left\{\frac{1}{\beta\left[-|\mathbf{k}|^2 + \omega^2/c_0^2 + 1\right]}\right\}\text{FFT}, \quad (6)$$

where FFT and IFFT are the forward and inverse fast Fourier transform operators. The Helmoltz equation can be solved iteratively provided that $\alpha \leq 1$ and $i\beta$ is a real number less than $\epsilon = \max\left|\frac{\omega^2}{c^2(\mathbf{x})} - \frac{\omega^2}{c_0^2}\right|$. The iteration can be truncated when the norm of the residual $\Gamma^{-1}(\mathbf{y} - A\mathbf{x}_n)$ falls below a predefined threshold (Swapnil and Vellekoop, 2024). To improve convergence, Vettenburg and Vellekoop (2023), suggest setting $\beta = -\frac{0.95i}{\epsilon}$ and $\alpha = 0.75$. An additional advantage of the proposed modified CBS is that the iterative solution defined by Eq. 6 requires only one fast convolution per iteration.

Results

To evaluate the performance of the proposed MBS method, we computed the wavefield and constructed the corresponding seismograms for a source located at the center of the surface ($\mathbf{x}_s$, red star in the velocity models), with receivers evenly distributed along the surface. Numerical experiments were conducted using two models of increasing complexity: a three-layer model (Figure 1) and the Marmousi model (Figure 2), with maximum frequencies of 50 Hz and 25 Hz, respectively. The stopping criterion for the MBS iterations was defined as a fixed number of steps: 120 for the three-layer model and 200 for the Marmousi model.

Seismograms for both models, shown in panels (b) and (c) of Figures 1 and 2, compare the REM and MBS. The NRMSE with respect to the REM seismograms was found to be relatively low: $E_{\text{3layer}} = 0.271$ and $E_{\text{Marmousi}} = 0.262$, indicating high accuracy. To further assess the similarity, we extracted a single time trace from the seismograms in both experiments (shown in panels (d) of Figures 1 and 2). The normalized traces display excellent agreement, with preserved phase alignment and minor amplitude differences. These slight discrepancies, likely due to boundary effects and iteration counts, suggest potential for further amplitude refinement.

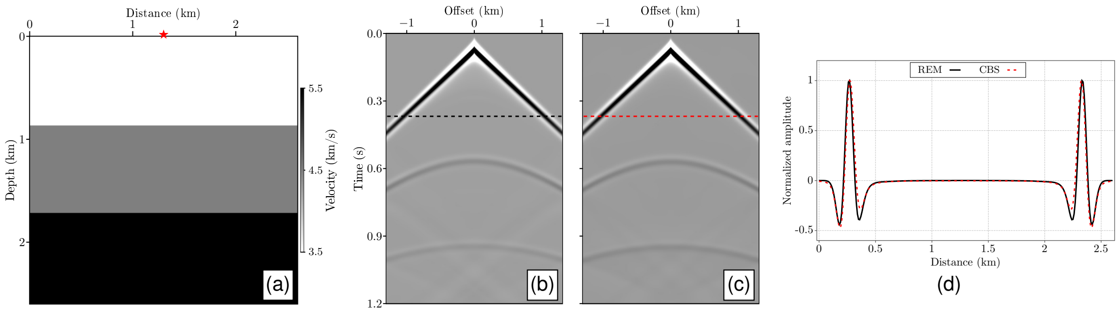

Figure 1: Three-layer model test: (a) velocity model, seismic responses obtained using (b) REM and (c) MBS methods, (d) comparison of normalized amplitude traces at $t = 0.37$ s.

Overall, the results demonstrate that the MBS method yields solutions that closely match those obtained with REM, confirming the accuracy and robustness of the proposed approach. While minor amplitude discrepancies were observed, likely due to boundary effects, these do not detract from the overall high performance. Future work will focus on analyzing MBS convergence criteria and computational performance compared to other Born series methods, aiming to enhance both efficiency and reliability in complex media.

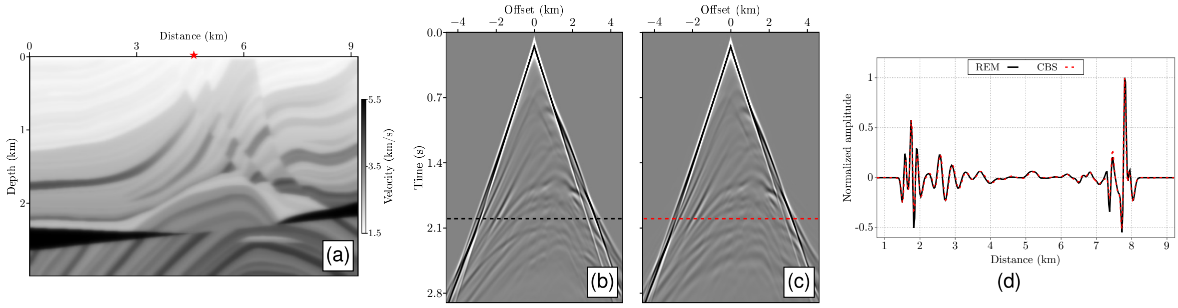

Figure 2: Marmousi model test: (a) velocity model, seismic responses obtained using (b) REM and (c) MBS methods, (d) comparison of normalized amplitude traces at $t = 2.00\,\mathrm{s}$.

Conclusions

The proposed MBS method offers a robust and efficient solution for the Lippmann-Schwinger equation in complex media with arbitrarily large structures and pronounced velocity contrasts. By integrating the scattering series into a preconditioned Richardson iteration, MBS ensures convergence in regimes where the TBS approach falters. Numerical comparisons with time-domain extrapolation validate its precision, demonstrating near-perfect alignment in seismogram amplitude and phase. Ongoing research will elucidate MBS's convergence behavior and computational efficiency relative to other scattering-based methods, further enhancing its applicability to intricate seismic modeling challenges.

Acknowledgments

This research was conducted within the framework of the Marchenko project, with financial support from CENPES/Petrobras. We would like to thank CENPES and SENAI CIMATEC for granting permission to publish this research. Reynam C. Pestana acknowledges financial support from CNPq through the INCT-GP.

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