On Some Formulas for Inverse Kinematic Problems in Ray Seismology

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ABSTRACT

The inverse kinematics problem of seismology in a plane is here considered in the general case, when a velocity of a propagation of wave depends on two variables. In addition, it is also considered the situations when a reflection, a refraction and caustics for a field of rays are present. For example, from the geophysical point of view, some of these results can be used for the analysis of velocity distribution between wells. Theoretically, one can obtain the mean value of the velocity, for example, for the earth's crust, but for this it is necessary a large amount of data, giving information of arrivals times, source-receiver coordinates, wave type, which requires a network of seismic stations and sources that is practically not simply reliable.

THEORECTICAL FOUNDATION: Travel-time and eikonal equations

Let D be a plane domain with a piecewise smooth boundary ∂D . In D, we consider an isotropic Riemannian metric

$$d\tau = n(x,y)ds\ , \qquad ds = \sqrt{dx^2 + dy^2}\ . \tag{1}$$

We denote by $k(l,z)$ the geodesic of the metric $d\tau$ with extremes l and z on the boundary. We choose the coordinate system on ∂D . For this, we choose an orientation (a direction) in ∂D , fix some point O on ∂D and for any point A on ∂D we define a distance l between this point A and point O . The number l is the coordinate of A. The number z is the coordinate of another point on ∂D . Let K be a family of all geodesics $k(l,z)$ of the metric $d\tau$ in $\overline{D} = D \cup \partial D$. Suppose that K satisfies the following two conditions: (a) any two points in $\overline{D}$ can be joined by a unique geodesic $k(l,z)$; (b) any geodesic $k(l,z)$ has exactly two points l and z on ∂D . Call such a family K of geodesics as regular. Note that the condition (a) follows a convexitivity of D relatively to a metric $d\tau$. In other words, for every $(l,z) \in \partial D \times \partial D$, a geodesic $k(l,z)$ exists. From condition (b) follows that in $\overline{D}$ the closed geodesics do not exist. An example of such domain with a metric $d\tau$ is in Figure 1. Another example is a circle with $n(x,y)$=constant, and consequently the geodesics are rectilinear segments. The length $\tau(l,z)$ of the geodesic $k(l,z)$ is expressed by the formula

$$\tau(l,z) = \int_{k(l,z)} n ds\ , \quad (l,z) \in \partial D \times \partial D\ ,\ k(l,z) \in K. \tag{2}$$

In this paper we considered the following inverse kinematics problem of seismology. Let the lengths $\tau(l,z)$ in (2) be known for all $(l,z) \in \partial D \times \partial D$. It is required to determine the metric (1), that is the function $n(x,y)$. In the mathematical aspect we have the following problem. We consider the equation (2) with respect to the function $n(x,y)$, $(x,y) \in \overline{D}$. In addition, also the geodesics $k(l,z)$ are unknown, and they are expressed by the unknown function $n(x,y)$. As a result we have a non-linear integral equation (2) relative to the function $n(x,y)$. In seismology, the function $\tau(l,z)$ is the propagation time of the perturbation from the source (an explosion, or an earthquake) located in a point l, to the receiver located in the point z. The function $n(x,y)$, slowness, is related to the velocity $v(x,y)$ of propagation of the particular wave type in the medium $\overline{D}$ by the formula $n=1/v$. In the case when D is a circle (Earth's section), and the function $n(x,y)$ is only depth dependent in the geophysical problem (2), the classical Herglotz-Wiechert's formula is applied. In our present case, the interest increases for problem (2), or other like problems, when the function $n(x,y)$ depends on both coordinate variables; therefore this situation is here considered. Geodesics is clearly a ray trajectory.
For problem (2) the uniqueness and the stability were obtained in reference [1]. We present briefly now this result. Let us consider two problems as in (2) in the same domain D:

$$\tau_i(l,z) = \int_{k_i(l,z)} n_i ds\ , \quad \text{for i=1,2, } (l,z) \in \partial D \times \partial D\ .$$

We denote $w(l,z) = \tau_1(l,z) - \tau_2(l,z)$.

Theorem 1: Stability. If $n_i(x,y) \in C^3(\overline{D})$, $\tau_i(l,z) \in C^1(\partial D \times \partial D)$, then for the solution of the problem (2) we have the stability estimate

$$\iint_D (n_1-n_2)^2 dxdy \le -\frac{1}{2\pi}\int_0^L\int_0^L \frac{\partial w}{\partial l}\frac{\partial w}{\partial z} dldz\,, \quad \text{where } L=\left|\partial D\right|. \tag{3}$$

For discussion on the proof of theorem 1, we introduce the function

$$\tau(x,y,z)=\int_{k(x,y,z)} nds\,, \quad \text{for } (x,y)\in D\,,\ z\in\partial D\,, \text{ or } (x,y,z)\in D\times\partial D\,,$$

where $k(x,y,z)$ is an arc of a geodesic which joins the points (x,y) and z. For fixed z, the function $\tau(x,y,z)$ is known in the calculus of variations as a field base function. It is known that the function $\tau(x,y,z)$ satisfies the eikonal equation

$$\tau_x^2+\tau_y^2=n^2(x,y)\,. \tag{4}$$

Now we write other necessary formulas for the function $\tau(x,y,z)$. Using a connection between vectors and covectors, we have that

$$\tau_x = n\cos\theta \quad \text{and} \quad \tau_y = n\sin\theta\,, \tag{5}$$

where $\theta=\theta(x,y,z)$ is an angle of a direction of geodesic $k(x,y,z)$ at the point (x,y). The following equality

$$\tau_x\cos\theta+\tau_y sin\theta = n(x,y) \tag{6}$$

is easily obtained from (5). From the properties of the field base function $\tau(x,y,z)$, the derivative of $\tau(x,y,z)$ along a ray front is equal to

$$\tau_x(-sin\theta)+\tau_y\cos\theta=0\,. \tag{7}$$

Differentiating the equality (4) with respect to z, we get

$$\frac{\partial}{\partial z}(\tau_x^2+\tau_y^2)=0\,, \quad \text{for } (x,y,z)\in D\times\partial D\,. \tag{8}$$

The function $\tau(x,y,z)$, when $(x,y)=l\in\partial D$, is the known function

$$\tau(l,z)\,, \quad (l,z)\in\partial D\times\partial D\,. \tag{9}$$

Equation (8) is a nonlinear hyperbolic-parabolic equation. It is easy to show that problem (2) is equivalent to the problem of determination of the function $\tau(x,y,z)$, satisfying equation (8) and a condition (9) on a solid torus $\partial D\times\partial D$. Calculation of (3) is performed by methods of energy estimations, using equation (8). The detailed proof of theorem 1 can be read in reference [1]. The analysis of the proof of the theorem 1 is now completed. At the present time it is not known, but it is possible to try to construct numerical methods for determination of an approximate solution for equation (8).

Theorem 2: Uniqueness. If $n(x,y)\in C^3(\overline{D})$, $\tau(l,z)\in C^1(\partial D\times\partial D)$, then the problem 2 can have only one solution.

Proof. The theorem 2 is the corollary of the theorem 1. Let in the theorem 1 $\tau_1(l,z)=\tau_2(l,z)$, then $w(l,z)=0$. In (3) the statement $n_1(x,y)=n_2(x,y)$ and theorem follows.

Using analogous methods as previously, we obtain the following theorem.

Theorem 3: Formula for the Riemannian volume in the regular case. If $n(x,y)\in C^3(\overline{D})$, $\tau(l,z)\in C^1(\partial D\times\partial D)$, then we have the following formula for the Riemannian volume (see Figure 1)

$$\iint_D n^2(x,y)dxdy=-\frac{1}{2\pi}\int_0^L\int_0^L\frac{\partial\tau}{\partial l}\frac{\partial\tau}{\partial z}dldz\,. \tag{10}$$

Poof. We rewrite (8) in the form of

$$L\tau\equiv\frac{\partial}{\partial z}(\tau_x\cos\theta+\tau_y sin\theta)=0\,. \tag{11}$$

In addition, we denote by L the differential operator in the left hand side of above equality (11), and reduce by the factor $n(x,y)$. The following identity is valid:

$$2(-\tau_x sin\theta+\tau_y\cos\theta)L\tau\equiv\theta_z(\tau_x^2+\tau_y^2)+\frac{\partial}{\partial z}[(\tau_x\cos\theta+\tau_y sin\theta)(\tau_y\cos\theta-\tau_x sin\theta)]-(\tau_x\tau_z)_y+(\tau_y\tau_z)_x=0\,. \tag{12}$$

To prove (12) it is sufficient to remove parentheses in the right hand side to get the left hand side. Zero in the second equality follows from (11). Substituting (7) in (12) we get $\partial/\partial z\,[...]=0$. Thus

$$\theta_z(\tau_x^2+\tau_y^2)-(\tau_x\tau_z)_y+(\tau_y\tau_z)_x=0\,. \tag{13}$$

Applying the Gauss–Ostrogradskii formula to (13), we get

$$\iiint_{D\times\partial D}(\tau_x^2+\tau_y^2)\theta_z dxdydz=-\int_0^L\int_0^L\tau_l\tau_z dldz\,.$$

Using (4) in the last equality, the formula (10) is obtained and the theorem 3 follows. Note that in (13) the first derivatives have integrable singularities in the neighborhood of the points (x,y)=z.

REFLECTION OF RAYS FROM A BOUNDARY

We consider now a situation when rays $k(x,y,z)$ are reflected from part of a boundary ∂D. Let D be a ring, i.e., a domain D enclosed between two circles $\partial_1 D=B_1$ and $\partial_2 D=B_2$ with radii r_1 and r_2, respectively, and $r_1>r_2$, $\partial D=\partial_1 D\cup\partial_2 D$. The metric $d\tau$ and the domain D must be such that there is a family K of geodesics, which we define below. Both ends

of every geodesic $k(l,z)$ belong to $\partial_1 D$. If a geodesic has a point on $\partial_2 D$, then it is reflected from $\partial_2 D$. As we consider the geodesics in every point (x,y) $\in D$ in all directions, then the family K has geodesics that are tangent to $\partial_2 D$. The geodesics that have no points on $\partial_2 D$ are smooth. Note that, if for a pair of points l and $z \in \partial_1 D$, a geodesic $k_1(l,z)$ exists and $k_1(l,z)$ is smooth, then with these extreme points l and z a geodesic $k_2(l,z)$, that has a point on $\partial_2 D$, at which it is reflected, exists, i.e. the points l and z $\in \partial_1 D$ join two geodesics $k_1(l,z)$ and $k_2(l,z)$. Certainly, exits pairs of points d and $f \in \partial_1 D$ with extremes d and f that the geodesic does not exist. It means that the closed domain $D \cup \partial D$ is not convex relatively to the geodesics of the metric $d\tau$. An example of such domain with a metric $d\tau$ is in Figure 2.
The Euclidean metric $d\tau^2 = dx^2 + dy^2$ and a ring D satisfy all enumerated conditions as in Figure 3. The example of Figure 2 is suggested from seismology. The domain D is the terrestrial crust, $\partial_1 D$ is the earth's surface and $\partial_2 D$ is the Moho surface, or the mantle interface. For domain D with a boundary $\partial D = \partial_1 D \cup \partial_2 D$ we have an analogous result as in theorem 3.

Theorem 4: Riemannian volume with reflection. If $n(x,y) \in C^3(\overline{D})$, $\tau_i(l,z) \in C^1(\partial D \times \partial D)$ for i=1 the function, $\tau_1(l,z)$ corresponds to a smooth geodesic $k_1(l,z)$ and, for i=2, $\tau_2(l,z)$ corresponds to a geodesic $k_2(l,z)$ with an incident ray and the reflected ray from $\partial_2 D$, then we have the formula of the Riemannian volume (see Figure 2)

$$\iint_D n^2(x,y)dxdy = \frac{1}{2\pi}\sum_{i=1}^{2}(-1)^i\int_0^L\sum_{j=1}^{2}(-1)^j\int_l^{p_j(l)}\frac{\partial\tau_i}{\partial z}\frac{\partial\tau_i}{\partial l}dzdl\,, \tag{14}$$

where l, $p(l)$ are the ends of the geodesics $k[l,p(l)]$ that is tangent $\partial_2 D$, $/\partial_1 D/ = L$.

Remark 1. In formula (14) we are able to consider a boundary $\partial_2 D$ as unknown, then a domain D is unknown. Thus, with an unknown domain D, and an unknown metric $d\tau = n(dx^2 + dy^2)^{1/2}$, with known times $\tau_i(l,z)$, i=1,2 for a propagation between the points l, $z \in \partial_1 D$ we are able to calculate the Riemannian volume for the domain D.

Remark 2. The formula (14) is valid also for a simply connected domain D with a boundary ∂D divided into two smooth parts $\partial_1 D$ and $\partial_2 D$, i.e., $\partial D = \partial_1 D \cup \partial_2 D$, and the boundaries of $\partial_1 D$ and $\partial_2 D$ must coincide; that is, $\partial\partial_1 D = \partial\partial_2 D$. In addition, as in theorem 4, every geodesic $k_2(l,z)$ has the incident ray and the reflected ray. (See Figure 4).

Proof. The theorem 4 is proved analogously to theorem 3 with the use of Snelliu's law. Here it is important to show that in (14) the integrals along the boundary $\partial_2 D$ and geodesics that are tangent to $\partial_2 D$ disappear. The formula (14) is found in reference [2].

FORMULAS FOR AN AVERAGE METRIC

From theorem 3 (formula 10) we are able to get a formula for [$n(x,y)$] for the mean square value of the metric $d\tau = n(dx^2 + dy^2)^{1/2}$ (for the function $n(x,y)$):

$$[n(x,y)] \overset{\text{def}}{=} \Big(\iint_D n^2(x,y)dxdy\Big)^{1/2} = |D|^{-1/2}\Big(\frac{1}{2\pi}\Big)^{1/2}\Big(-\int_0^L\int_0^L\frac{\partial\tau_i}{\partial z}\frac{\partial\tau_i}{\partial l}dldz\Big)^{1/2}. \tag{15}$$

where $/D/$ is an Euclidean area of the domain D. If in the formula (14), in addition an area $/D/$ is known, then an analogous formula to (15) is obtained, which is :

$$[n(x,y)] \overset{\text{def}}{=} \Big(\iint_D n^2(x,y)dxdy\Big)^{1/2} = |D|^{-1/2}\Big(\frac{1}{2\pi}\Big)^{1/2}\Big(\sum_{i=1}^{2}(-1)^i\int_0^L\sum_{j=1}^{2}(-1)^j\int_l^{p_j(l)}\frac{\partial\tau_i}{\partial z}\frac{\partial\tau_i}{\partial l}dzdl\Big)^{1/2}. \tag{16}$$

APPLICATION TO WELL-LOG SEISMIC TOMOGRAPHY

We consider the above results to a well-log geophysical application for the analysis of the medium between drilled wells. All the stated results are valid for the domain D with a piecewise smooth boundary. The domain D is convex relative to its metric. Let D be a rectangle with the sides a, b, c and d. The side a corresponds to the earth's surface, b and d are the lateral sides (drilled wells), and side c is the base of the considered domain. We enumerate below source-receiver arrays for the experiments in different problems, where the obtained above results apply.

Problem 1. There are sources and receivers in every point of the boundary $\partial D = a \cup b \cup c \cup d$.

Problem 2. There are sources and receivers in every point of the sides $\partial D = a \cup b \cup c$, and only receivers in side d.

Problem 3. There are sources and receivers in every point of the sides $\partial D = a \cup b$, and only sources in side c and receivers in side d.

Problem 4. There are sources and receivers in every point of the sides $\partial D = a \cup b$, and only sources in side d and receivers in side c.

Problem 5. There are sources and receivers in every point of the side a, and only sources in side d and receivers in side b, and from side c the geodesics are reflected. (See Figure 5).

Let the domain D be a triangle with the sides a, b and c. The side a corresponds to the earth's surface.

Problem 6. There are sources and receivers on side *a*, and only sources on side *b* and receivers on side *c*.

CONCLUSIONS

The obtained results are to be used in the following way. For problems 1-6 we obtain the mean metric, i.e. the function *n*(*x*,*y*). With this information, together with some other geophysical complementary information, it is then possible to arrive at some conclusions on the metric $d\tau$ of the domain *D*.

Analogous results in R^3 exist in the case of the Riemannian metric, where the obtained formulas are generalizations in the case of refraction of the rays and in the case of presence of caustics.

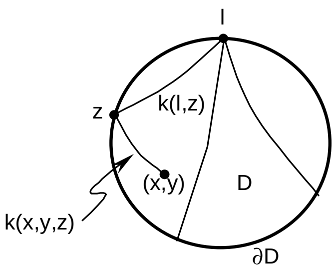

Figure 1. Regular case. Earth's section without nucleus.

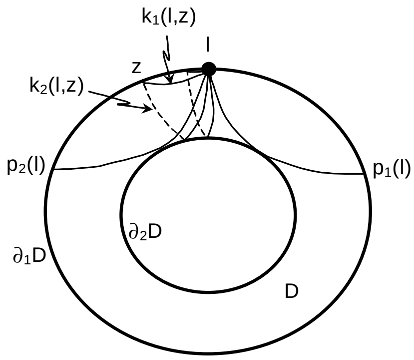

Figure 2. Internal refraction and reflection in D only.

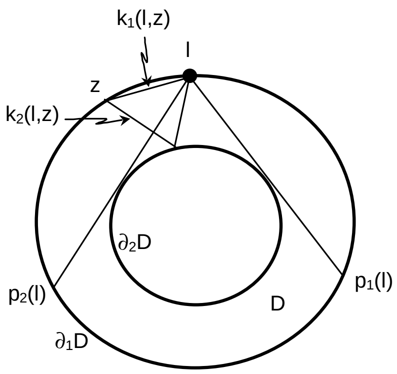

Figure 3. Constant velocity Earth.

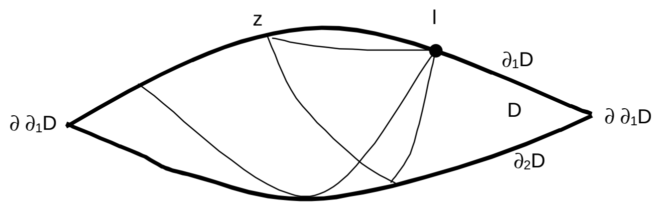

Figure 4. Simple tomography structure with 2 surfaces.

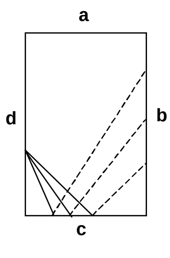

Figure 5. Well-log seismic tomography with 4 sides: earth's surface, 2 drilled wells and formation discontinuity.

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