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The Kirchhoff-Helmholtz transform pair

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Abstract

Modeling a reflected wave by the Kirchhoff-Helmholtz integral consists of an integration along the reflector. By this, one sums up the Huygens secondary-source contributions to the wavefield at the observation point. The proposed asymptotic inverse Kirchhoff-Helmholtz integral, by which this modeling process is inverted, works in a completely analogous way. It consists of an integral along the reflection traveltime surface of the reflector. For a point on the reflector, one sums up the reflected-wave contributions present at the respective reflection-traveltime surface associated with the related source-receiver pair. The new inverse integral reconstructs the Huygens sources along the reflector, providing their positions and amplitudes. In this way, one can devise a new true-amplitude migration algorithm.

INTRODUCTION

The wavefield originating from a point source and primarily reflected from a smooth reflector overlain by a smooth inhomogeneous acoustic medium can be described by the Kirchhoff integral in the so-called single-scattering, high-frequency approximation (see, e.g., Frazer and Sen, 1985). The resulting Kirchhoff-Helmholtz integral describes then the reflected elementary waves as a superposition of Huygens secondary point sources distributed along the reflector.

The Kirchhoff-Helmholtz integral is largely used to accurately model primary reflections in smoth layered models bounded by smooth interfaces (reflectors). A natural question that arises is whether a transformation exists that performs the opposite task of the Kirchhoff-Helmholtz integral. In other words, this inverse would have to *kinematically and dynamically reconstruct* the reflector. This would have to involve a weighted superposition of the observed elementary wave along the reflection traveltime surface of the searched-for reflector. To kinematically and dynamically reconstruct the reflector means to asymptotically recover the reflector location together with the plane-wave reflection coefficient in each point of the reflector. In the seismic literature, this is commonly called the *true amplitude* at all reflector points.

The depth migration method traditionally accepted as an inverse to the Kirchhoff-Helmholtz integral is Kirchhoff depth migration (Schneider, 1978). This migration is realized upon summing up contributions of the reflection data along auxiliary diffraction surfaces constructed on an a priori given reference model.

We see that the Kirchhoff-Helmholtz integral, a summation operator along a given reflector, lacks a structurely similar (asymptotic) inverse operation. This should have the form of a summation operation along the reflection traveltime corresponding the reflector, assuming, of course, the same configuration of source-receiver pairs. This is being set up in this paper by exploring the dual properties between the given reflector and its corresponding traveltime surface.

FORMULATION OF THE PROBLEM

The Kirchhoff-Helmholtz integral transformation pair is based on the following assumptions.

(1) Referring to Figure 1, we assume the model of a smoothly varying inhomogeneous acoustic medium, bounded above by the measurement surface, $z = 0$, and below by the target reflector Σ. The reflector Σ is, for simplicity, assumed to be parameterized as $z = \Sigma(\vec{x})$, in which $\vec{x}$ is the two-dimensional horizontal coordinate vector varying on the spatial aperture set E. Points on the reflector Σ will be generally denoted by $M_\Sigma = (\vec{x}, z = \Sigma(\vec{x}))$. (2) The locations of the source-receiver pairs (S, G) are given as a function of a two-dimensional vector parameter $\vec{\xi}$ that varies on a given configuration aperture set A. For each source-receiver pair, we suppose that there exists exactly one point $M_R = (\vec{x}_R, \Sigma(\vec{x}_R))$ on the reflector Σ, for which the composite ray SM_RG describes a specular primary reflection. The dependency $\vec{x}_R = \vec{x}_R(\vec{\xi})$ implies that the location of the specular reflection point M_R is determined by the location of the source-receiver pair (S, G) specified by $\vec{\xi}$. We will denote by $\mathcal{R}(M_R)$ the plane-wave reflection coefficient for the ray SM_RG at M_R. (3) The function $t = \Gamma(\vec{\xi})$ describes the reflection traveltime from the source $S(\vec{\xi})$ to the receiver $G(\vec{\xi})$ along the primary-reflection ray SM_RG. This function is called the reflection-traveltime surface Γ of the target reflector Σ. Both surfaces are said to be dual of each other. Points on the traveltime surface Γ will be denoted by $N_\Gamma = (\vec{\xi}, t = \Gamma(\vec{\xi}))$. (4) For each point M_Σ on the reflector Σ, we correspondingly assume that there exists exactly one source-receiver pair (S_R, G_R) for which the composite ray $S_R M_\Sigma G_R$ pertains to a specular primary reflection at M_Σ. This pair (S_R, G_R) is parameterized by a fixed value of $\vec{\xi}_R = \vec{\xi}_R(\vec{x})$

depending on the horizontal coordinate $\vec{x}$ of M_Σ. We will denote by $\mathcal{R}(M_\Sigma)$ the plane-wave reflection coefficient for the ray $S_R M_\Sigma G_R$ at M_Σ. Also, the notation $N_R = (\vec{\xi}_R, \Gamma(\vec{\xi}_R))$ will be used for a point on Γ pertaining to $\vec{\xi}_R$, i.e., to the fixed source-receiver pair (S_R, G_R). (5) At any specified point S, on the measurement surface, we will consider an exploding point source with a impulse-shaped source signal $\delta(t)$. The effects of a limited bandwidth do not influence the present analysis and need not be considered. (6) Finally, we assume reproducible point sources of unit strength and an omnidirectional radiation pattern. We also neglect the transmission loss due to interfaces in the overburden. In addition, all other factors affecting the seismic amplitudes apart from geometrical spreading are assumed to be negligible or have been corrected for.

Under these assumptions, zero-order ray-theory provides the following description of a primary reflected elementary wave. For each source-receiver pair (S, G), the reflection event at the receiver is described by

$$K_\Gamma(\vec{\xi}, t) = \mathcal{R}(M_R)/\mathcal{L} \;\; \delta(t - \Gamma(\vec{\xi})) \; . \tag{1}$$

In the above formula, the amplitude factors $\mathcal{R}(M_R)$ and $\mathcal{L}$ are the plane-wave reflection coefficient at M_R and the geometrical-spreading factor pertaining to the specular reflection ray SM_RG. Note again that each point N_Γ on Γ is associated with exactly one point M_R on Σ.

We see that $K_\Gamma(\vec{\xi}, t)$ is aligned along the reflection-traveltime surface Γ as defined above. We may say that $K_\Gamma(\vec{\xi}, t)$ is the image of the reflector Σ at the reflection-traveltime surface Γ in the time domain. In other words, the image $K_\Gamma(\vec{\xi}, t)$ describes what we can observe about the reflector in the recorded reflected wavefield.

We next introduce the function $I_\Sigma(\vec{x}, z)$, which is aligned along the target reflector Σ. For each $\vec{x}$ in E and all real z, the function $I_\Sigma(\vec{x}, z)$ is defined by

$$I_\Sigma(\vec{x}, z) = \mathcal{R}(M_\Sigma)\, \delta(z - \Sigma(\vec{x})) \; , \tag{2}$$

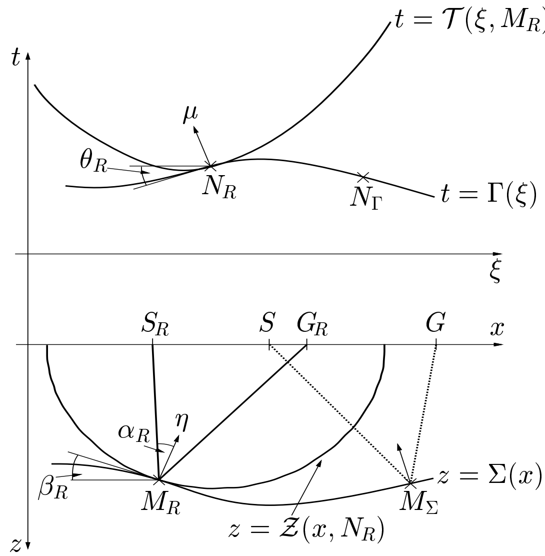

Figure 1: One-interface model together with auxiliary curves: isochrones and diffraction curves.

Note that the function $I_\Sigma(\vec{x}, z)$ is the complex version of the *singular function of the reflector* as introduced by Bleistein (1987). It is defined here, however, in a true-amplitude sense, i.e., with the varying reflection coefficient along the reflector as its amplitude. Moreover, in the same way as expression (2) is referred to as the analytic singular function of the reflector, we can interpret equation (1) as the analytic singular function of the reflection traveltime surface.

DIFFRACTION TRAVELTIMES AND SPATIAL ISOCHRONES

For arbitrary vector parameters $\vec{\xi}$ in A and arbitrary subsurface points $M = (\vec{x}, z)$, we introduce the *diffraction traveltime* surface

$$t = \mathcal{T}(\vec{\xi}, \vec{x}, z) = \mathcal{T}(\vec{\xi}, M) = T(S(\vec{\xi}), M) + T(G(\vec{\xi}), M) \; , \tag{3}$$

namely the sum of traveltimes from the source-receiver pair (S, G) specified by $\vec{\xi}$ to the subsurface point M. The above formula expresses the traveltime from the *diffractor point* M to the source-receiver pair (S, G).

The reflection traveltime surface $t = \Gamma(\vec{\xi})$ of the given refletor Σ can be, as a consequence, recast as

$$t = \Gamma(\vec{\xi}) = \mathcal{T}(\vec{\xi}, M_R) = \mathcal{T}(\vec{\xi}, \vec{x}_R, \Sigma(\vec{x}_R)) \; , \tag{4}$$

where the horizontal vector coordinate $\vec{x}_R = \vec{x}_R(\vec{\xi})$ locates the reflection point $M_R = (\vec{x}_R, \Sigma(\vec{x}_R))$ on Σ determined by the source-receiver pair specified by $\vec{\xi}$.

We next consider the spatial counterparts of the traveltime functions defined above. For any $\vec{x}$ in E and arbitrary points $N = (\vec{\xi}, t)$ in the time domain, we introduce the *isochrone* $z = \mathcal{Z}(\vec{x}, \vec{\xi}, t)$ as the locus of points $M_\mathcal{Z} = (\vec{x}, z = \mathcal{Z}(\vec{x}, \vec{\xi}, t))$ for which the diffraction traveltime to a fixed source-receiver pair equal the given traveltime t, viz.

$$\mathcal{T}(\vec{\xi}, M_\mathcal{Z}) = T(S(\vec{\xi}), M_\mathcal{Z}) + T(G(\vec{\xi}), M_\mathcal{Z}) = t \; . \tag{5}$$

An important observation is that the reflector function $z = \Sigma(\vec{x})$ can be recast as a restriction of the above isochrone functions, namely

$$z = \Sigma(\vec{x}) = \mathcal{Z}(\vec{x}, \vec{\xi}_R, \Gamma(\vec{\xi}_R)) \; , \tag{6}$$

where $\vec{\xi}_R = \vec{\xi}_R(\vec{x})$ is the vector parameter that specifies the source-receiver pair S_R and G_R for which the two ray segments $S_R M_\Sigma$ and $M_\Sigma G_R$ constitute a reflection ray.

Diffraction traveltime surfaces $t = \mathcal{T}(\vec{\xi}, \vec{x}, \Sigma(\vec{x}))$, for fixed $\vec{x}$, and isochrone surfaces $z = \mathcal{Z}(\vec{x}, \vec{\xi}, \Gamma(\vec{\xi}))$ for fixed $\vec{\xi}$ are connected by duality relationships. The diffraction-traveltime surface for a point M_Σ is tangent to Γ at a point N_Γ. Correspondingly, the isochrone surface for N_Γ on Γ is tangent to Σ at M_Σ. Also, further relationships between the dips and curvatures of Σ and Γ in M_Σ and N_Γ, respectively, can be established.

THE KIRCHHOFF-HELMHOLTZ INTEGRAL PAIR

The Kirchhoff-Helmholtz (KH) modeling integral, called from now on the forward KH integral, asymptotically computes the singular function $K_\Gamma(\vec{\xi},t)$ of the reflection-traveltime surface Γ. Input to this calculation are the location of the reflector Σ, the velocity distribution in the reflector overburden, and the values of the reflection coefficient $\mathcal{R}(M_\Sigma)$ along Σ.

In a completely analogous way, the inverse KH integral to be defined below, asymptotically computes the singular function $I_\Sigma(\vec{x},z)$ of the reflector Σ. Input to this calculation are the location of the reflection-traveltime surface Γ, the velocity distribution of the reflector overburden, and the wavefield amplitude $\mathcal{R}(M_R)/\mathcal{L}$ along Γ.

Under the assumptions stated in the previous section, the forward KH integral can be written as an integral along the reflector Σ in the form (Frazer and Sen, 1985; Tygel et al., 1994)

$$K(\vec{\xi},t) = \frac{1}{4\pi}\int d\Sigma\, \mathcal{W}_K(\vec{\xi},M_\Sigma)\; \mathcal{R}(M_\Sigma)\; \partial_\eta \delta(t-\mathcal{T}(\vec{\xi},M_\Sigma))\;, \qquad (7)$$

where $K(\vec{\xi},t)$ is the modeled elementary wave at the receiver $G(\vec{\xi})$. Also, ∂_η denotes the partial derivative in the direction of the normal to the surface Σ at M_Σ. Under the above-mentioned assumption that transmission losses in the overburden can be neglected, the weight function is given by

$$\mathcal{W}_K(\vec{\xi},M_\Sigma) = 1/\mathcal{L}_S\,\mathcal{L}_G\;, \qquad (8)$$

where $\mathcal{L}_S$ and $\mathcal{L}_G$ denote the geometrical-spreading factors along the two ray branches from the source S to the point M_Σ and from there to the receiver G (see Figure 1).

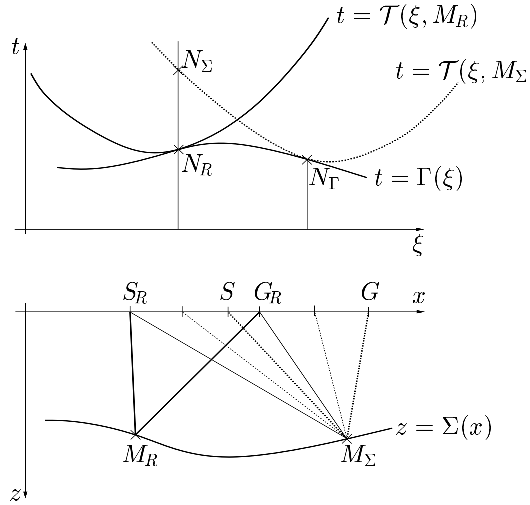

Figure 2: The forward Kirchhoff-Helmholtz integral understood geometrically. For each point M_Σ on Σ, the integration contributes to the reflection response computed for $\vec{\xi}_R$ at the corresponding point $N_\Sigma = (\vec{\xi}_R, \mathcal{T}(\vec{\xi}_R, M_\Sigma))$. For details see text.

Let us now investigate integral (7) more closely in order to better understand it geometrically. This will help us to set up an analogous integral for its inversion. For the following discussion, we refer to Figure 2.

We start by considering a certain, fixed value $\vec{\xi}_R$, where we want to compute the reflected wave as a function of time. We denote by M_R the reflection point on Σ which corresponds to the source-receiver pair (S_R, G_R) defined by $\vec{\xi}_R$. The point M_R defines a diffraction traveltime surface that is tangent to the reflection traveltime surface Γ at a point $N_R = (\vec{\xi}_R, \Gamma(\vec{\xi}_R))$, called the dual to point M_R. For each point M_Σ on the reflector, integral (7) contributes to the final response $K(\vec{\xi}_R,t)$ at a single point $N_\Sigma = (\vec{\xi}_R, \mathcal{T}(\vec{\xi}_R, M_\Sigma))$, where $\mathcal{T}$ is defined in equation (3). In other words, N_Σ is the point where the diffraction traveltime surface $t = \mathcal{T}(\vec{\xi}, M_\Sigma)$, cuts the vertical line at $\vec{\xi}_R$ (see Figure 2). The point N_Σ will fall onto Γ, i.e., it will coincide with point N_R, the dual point to M_R, when M_Σ coincides with M_R. At N_R, the diffraction traveltime surface of M_R, $t = \mathcal{T}(\vec{\xi}, M_R)$, is tangent to Γ. We thus have a stationary situation at N_R, which means that the main contribution of integral (7) will be observed at that point. In other words, the forward KH integral (7) transforms the singular function of reflector Σ into its image at Γ. The weight function $\mathcal{W}_K(\vec{\xi}, M_\Sigma)$ serves to perform this transformation in a dynamically correct way, i.e., yielding the correct wave amplitude and pulse shape at N_R.

To set up a completely analogous integral that achieves the inverse task, namely to reconstruct the singular function of the reflector Σ from its image at Γ, we only have to substitute in the above integral all points and surfaces by their respective duals. This is geometrically described with the help of Figure 3.

The new integral will consist of an integration along the reflection-traveltime surface Γ instead of the reflector Σ. In analogy with the preceding construction, we consider the output of the integration at a certain, fixed coordinate $\vec{x}_R$ which determines a (supposedly unique) dual point $N_R = (\vec{\xi}_R, \Gamma(\vec{\xi}_R))$ on the reflection traveltime surface Γ. The isochrone specified by the point N_R will be tangent to the reflector Σ at the (supposedly unique) point $M_R = (\vec{x}_R, \Sigma(\vec{x}_R))$. For each point N_Γ on Γ, the new integral has to contribute to the final result $I(\vec{x}_R, z)$ at a certain point M_Γ. This point M_Γ should be located at the position where the isochrone of N_Γ, $z = \mathcal{Z}(\vec{x}, N_\Gamma)$ cuts the vertical line at $\vec{x}_R$, i.e., $M_\Gamma = (\vec{x}_R, \mathcal{Z}(\vec{x}_R, N_\Gamma))$.

The point M_Γ will fall on Σ, i.e., it will coincide with M_R, when N_Γ coincides with N_R, the dual point of M_R. At M_R, the isochrone $z = \mathcal{Z}(\vec{x}, N_R)$ is tangent to Σ. Due to our above assumption of a smooth reflector and uniqueness of dual points, we have again the situation of an isolated singularity at M_R, which means that the main contribution of the new integral will be observed at M_R. In this way, we have geometrically constructed a transformation of the reflection-traveltime function Σ into the reflector Σ. A certain weight function will be included into the integral in order to assure that also this inverse transformation can be performed in a dynamically correct way, i.e., to correctly reconstruct the varying reflection coefficient along the reflector Σ.

Translating the above observations in mathematical terms and in full correspondence to the forward KH integral, we can now set up the following inverse KH integral,

$$I(\vec{x},z) = -\int d\Gamma\; \mathcal{W}_I(\vec{x},N_\Gamma)\; \frac{\mathcal{R}(M_R)}{4\pi\mathcal{L}}\; \partial_\mu \delta(z-\mathcal{Z}(\vec{x},N_\Gamma))\;. \qquad (9)$$

where $I(\vec{x}, z)$ is the final imaging result. In this formula, ∂_μ denotes, correspondingly to ∂_η above, the partial derivative in the direction of the normal to the traveltime surface Γ at N_Γ. We recall that M_R is the specular reflection point on the reflector pertaining to the source-receiver pair (S, G) defined by $\vec{\xi}$. From the asymptotic analysis of integral (9), the weight function results as

$$\mathcal{W}_I(\vec{x}, N_\Gamma) = h_B \; v^3 \; \cos^2\theta \; \mathcal{L}_S \mathcal{L}_G / \cos^2\alpha \; , \tag{10}$$

where θ represents the "local dip angle" of the reflection-traveltime surface Γ (i.e., the angle the normal to Γ at N_Γ makes with the vertical t-axis), and α denotes the incidence angle the incoming ray-branch slowness vector makes with the isochrone normal at M (see Figure 3). Moreover, h_B is the modulus of the Beylkin determinant (Bleistein, 1987). All these quantities are computed for the actual point N_Γ.

In mathematical terms, the two stationary situations mentioned above relate to the following statements for the asymptotic integral results. As is well known (see, for example, Tygel et al., 1994) the Kirchhoff-Helmholtz integral (7) can be approximated, for time values $t \approx \Gamma(\vec{\xi})$, by the zero-order ray-theoretical expression, viz.,

$$K(\vec{\xi}, t) \approx K_\Gamma(\vec{\xi}, t) = \mathcal{R}(M_R)/\mathcal{L}(\vec{\xi}) \;\; \delta(t - \Gamma(\vec{\xi})) \; . \tag{11}$$

As indicated above, correspondingly, the approximation of integral (9) for $z \approx \Sigma(\vec{x})$ yields in an asymptotic sense

$$I(\vec{x}, z) \approx I_\Sigma(\vec{x}, z) = \mathcal{R}(M_\Sigma) \; \delta(z - \Sigma(\vec{x})) \; , \tag{12}$$

i.e., the (complex) singular function of the reflector. This means that integral (9) is the inverse to the forward KH integral (7), or, in other words, integrals (7) and (9) form a transform pair between the depth-domain image $I_\Sigma(\vec{x}, z)$ of the target reflector and its time-domain image $K_\Gamma(\vec{\xi}, t)$ in multi-coverage reflection data.

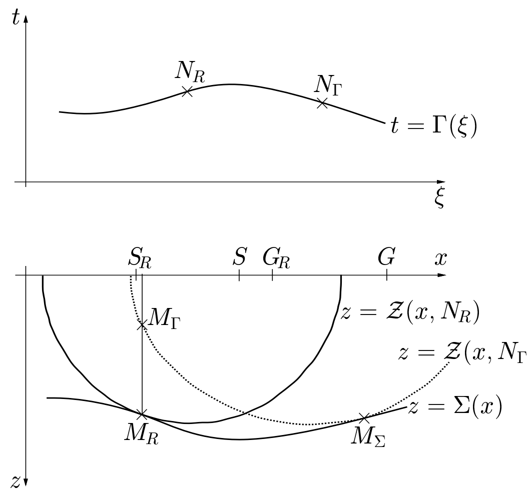

Figure 3: The inverse Kirchhoff-Helmholtz integral understood geometrically. For each point N_Γ on Γ, the integration contibutes to the reflector depth image computed for $\vec{x}_R$ at the corresponding point $M_\Gamma = (\vec{x}_R, \mathcal{Z}(\vec{x}_R, N_\Gamma))$. For details see text.

CONCLUSIONS

Based on the duality between reflectors and reflection time surfaces, we have presented a new inverse Kirchhoff-Helmholtz integral that is completely analogous to the forward integral. The inverse Kirchhoff-Helmholtz integral fills a recently discovered gap which originates from the observation that the conventional Kirchhoff migration integral (Schneider, 1978), well known in the seismic literature, is not an inverse to the forward Kirchhoff-Helmholtz integral. The proposed inverse Kirchhoff-Helmholtz integral enables the design of a new seismic migration technique that would deserve the name Kirchhoff migration much more than what is up to now associated with this name. The construction of true-amplitude migrated reflector images by the new migration technique can be achieved by the superposition of their elementary reflection images along the reflection-traveltime surface. In this way, the migration can be realized as a weighted stack along the (identified and picked) reflection-time surface instead of stacking along the conventional diffraction-time surfaces (computed on a whole region in depth). Of course, to recover the correct reflector position as well as to calculate the weight function, both techniques require an a priori given velocity model of the reflector overburden.

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