Reciprocity theorems for time-lapse seismics based on the full and one-way wave equations
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Abstract

In this paper we present acoustic reciprocity theorems for the full and one-way wave equations and we discuss their application in time-lapse seismics.

INTRODUCTION

Reciprocity theorems play an important role in formulating true amplitude operations on seismic wave fields, such as multiple elimination, migration and characterization. In general, a reciprocity theorem interrelates the quantities that characterize two admissible physical states that could occur in one and the same domain (de Hoop, 1988). One state is identified with an actual measurement, while the other state can either be a computational state (e.g. migration operators), a desired state (e.g. multiple-free data) or an other measurement (characterizing time-lapse differences in the reservoir).
In the usual practice of seismic data analysis two classes of wave equations are used, viz. the full wave equation expressed in terms of the acoustic pressure and particle velocity and the one-way wave equations expressed in terms of down and up going waves. Accordingly, reciprocity theorems can be formulated for both classes of wave equations. In this paper we present reciprocity theorems for the full wave field as well as for its down and up going constituents and we discuss some of their applications.

Reciprocity theorem for the full wave field

In this section we review the scalar form of the acoustic reciprocity theorem of the convolution type. We closely follow de Hoop (1988) and Fokkema and van den Berg (1993). The former author derives reciprocity theorems in the time domain; the latter authors in the time domain, the Laplace domain and the frequency domain. Here we only consider the frequency domain.

Basic acoustic equations

In the space-frequency $(\mathbf{x},\omega)$ domain, the equations that govern linear acoustic wave motion read

$$\partial_k P + j\omega\varrho V_k = F_k \quad \text{and} \quad \partial_k V_k + j\omega\kappa P = Q, \tag{1}$$

where P is the acoustic pressure, V_k is the particle velocity, ϱ is the volume density of mass, κ is the compressibility, F_k is the volume source density of volume force and Q is the volume source density of volume injection rate. The Latin subscripts take on the values 1 to 3 and the summation convention applies to repeated subscripts.

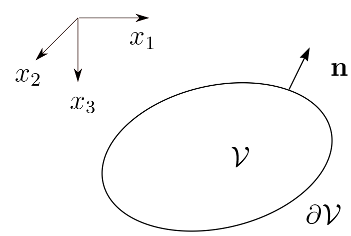

Figure 1: Configuration for Rayleigh's reciprocity theorem.

We introduce two acoustic states (i.e., wave fields, medium parameters and sources), that will be distinguished by the subscripts A and B. For these two states we consider the interaction quantity $\partial_k\{P_A V_{k,B} - V_{k,A} P_B\}$. Applying the product rule for differentiation, substituting equations (1) for states A and B, integrating the result over a volume $\mathcal{V}$ with boundary $\partial\mathcal{V}$ and outward pointing normal vector $\mathbf{n} = (n_1, n_2, n_3)$ (see Figure 1) and applying the theorem of Gauss yields

$$\int_{\mathbf{x}\in\partial\mathcal{V}} \{P_A V_{k,B} - V_{k,A} P_B\} n_k \mathrm{dA} = -j\omega \int_{\mathbf{x}\in\mathcal{V}} \{P_A(\kappa_B - \kappa_A) P_B - V_{k,A}(\varrho_B - \varrho_A) V_{k,B}\} \mathrm{dV} + \int_{\mathbf{x}\in\mathcal{V}} \{P_A Q_B - V_{k,A} F_{k,B} + F_{k,A} V_{k,B} - Q_A P_B\} \mathrm{dV}. \tag{2}$$

Equation (2) is Rayleigh's reciprocity theorem (Rayleigh, 1878).
We conclude this section by considering some special cases.
Unbounded media – Consider the situation in which the medium at and outside $\partial\mathcal{V}$ is homogeneous, unbounded and source-free in both states. Assume that the wave fields in both states are causally related to the sources in $\mathcal{V}$. Then, if $\varrho_A = \varrho_B$ and $\kappa_A = \kappa_B$ at and outside $\partial\mathcal{V}$, the boundary integral on the left-hand side of equation (2) vanishes (Bleistein, 1984; Fokkema and van den Berg, 1993).
Physical reciprocity – Assume that the above mentioned conditions are fulfilled and that $\varrho_A = \varrho_B$ and $\kappa_A = \kappa_B$ in $\mathcal{V}$ as well. Then the first volume integral on the right-hand side of equation (2) vanishes. Furthermore, consider point sources in states A and B at $\mathbf{x}_A \in \mathcal{V}$ and $\mathbf{x}_B \in \mathcal{V}$, respectively, according to $Q_A(\mathbf{x},\omega) = q_A(\omega)\delta(\mathbf{x}-\mathbf{x}_A)$, $Q_B(\mathbf{x},\omega) = q_B(\omega)\delta(\mathbf{x}-\mathbf{x}_B)$, with $q_A(\omega) = q_B(\omega)$ and $F_{k,A}(\mathbf{x},\omega) = F_{k,B}(\mathbf{x},\omega) = 0$. Equation (2) thus yields the well-known result

$$P_A(\mathbf{x}_B|\mathbf{x}_A,\omega) = P_B(\mathbf{x}_A|\mathbf{x}_B,\omega). \tag{3}$$

Reciprocity theorem for one-way wave fields

In this section we review the matrix-vector form of the acoustic reciprocity theorem for one-way wave fields (Wapenaar and Grimbergen, 1996).

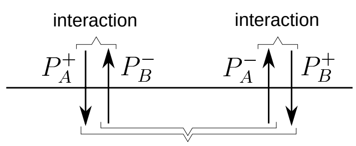

Figure 2: Both terms of the interaction quantity for the one-way reciprocity theorem of the convolution type contain waves that propagate in opposite directions.

We introduce a system of coupled equations for the one-way wave fields P^+ and P^-, propagating in the positive and negative depth direction, respectively, originating from sources S^+ and S^-:

$$\partial_3 \mathbf{P} = \hat{\underset{\sim}{\mathbf{B}}}\mathbf{P} + \mathbf{S} \tag{4}$$

(the hat denotes a pseudo-differential operator), with

$$\mathbf{P} = \begin{pmatrix} P^+ \\ P^- \end{pmatrix}, \mathbf{S} = \begin{pmatrix} S^+ \\ S^- \end{pmatrix} \quad \text{and} \quad \hat{\underset{\sim}{\mathbf{B}}} = \begin{pmatrix} -j\hat{\mathcal{H}}_1 & 0 \\ 0 & j\hat{\mathcal{H}}_1 \end{pmatrix} + \begin{pmatrix} \hat{T} & -\hat{R} \\ -\hat{R} & \hat{T} \end{pmatrix}, \tag{5}$$

in which $\hat{\underset{\sim}{\mathbf{B}}}$ is the one-way operator matrix, $\hat{\mathcal{H}}_1$ is the well-known square-root operator, and $\hat{R}$ and $\hat{T}$ are the reflection and transmission operators, respectively.
We introduce two different states that will be distinguished by the subscripts A and B. For these two states we consider the interaction quantity $\partial_3\{\mathbf{P}_A^T \underset{\sim}{\mathbf{N}} \mathbf{P}_B\}$, with $\underset{\sim}{\mathbf{N}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ or, written alternatively, $\partial_3\{P_A^+ P_B^- - P_A^- P_B^+\}$. Apparently, we consider the interaction between oppositely propagating waves (see Figure 2).
Applying the product rule for differentiation, substituting the one-way wave equation (4) for states A and B, integrating the result over a cylindrical volume $\mathcal{V}$ with boundary $\partial\mathcal{V}_0 \cup \partial\mathcal{V}_1$ (see Figure 3), applying the theorem of Gauss and using the symmetry relation $\hat{\underset{\sim}{\mathbf{B}}}^\dagger \iff -\underset{\sim}{\mathbf{N}}\hat{\underset{\sim}{\mathbf{B}}}^* \underset{\sim}{\mathbf{N}}^{-1}$, yields the following one-way reciprocity theorem

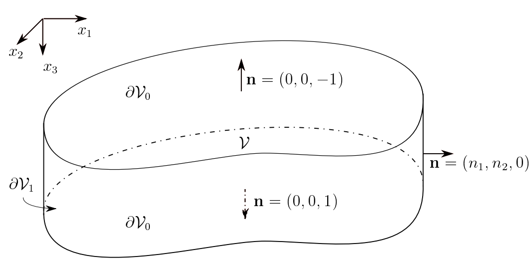

Figure 3: Modified configuration for the one-way reciprocity theorem. The combination of the two planar surfaces is denoted by $\partial\mathcal{V}_0$; the cylindrical surface is denoted by $\partial\mathcal{V}_1$.

$$\begin{aligned} \int_{\mathbf{x}\in\partial\mathcal{V}_0} \mathbf{P}_A^T \underset{\sim}{\mathbf{N}} \mathbf{P}_B n_3 \mathrm{dA} &= \int_{\mathbf{x}\in\mathcal{V}} \mathbf{P}_A^T \underset{\sim}{\mathbf{N}} \hat{\underset{\sim}{\mathbf{\Delta}}} \mathbf{P}_B \mathrm{dV} \\ &+ \int_{\mathbf{x}\in\mathcal{V}} \{\mathbf{P}_A^T \underset{\sim}{\mathbf{N}} \mathbf{S}_B + \mathbf{S}_A^T \underset{\sim}{\mathbf{N}} \mathbf{P}_B\} \mathrm{dV}, \end{aligned} \tag{6}$$

where the contrast operator $\hat{\underset{\sim}{\mathbf{\Delta}}}$ is given by

$$\hat{\underset{\sim}{\mathbf{\Delta}}} = \hat{\underset{\sim}{\mathbf{B}}}_B - \hat{\underset{\sim}{\mathbf{B}}}_A. \tag{7}$$

Note that the boundary integral over $\partial\mathcal{V}_1$ vanished. For bounded $\partial\mathcal{V}_1$ this occurs when $\mathbf{P}_A$ and $\mathbf{P}_B$ satisfy homogeneous Dirichlet or Neumann boundary conditions on $\partial\mathcal{V}_1$. On the other hand, when $\partial\mathcal{V}_1$ is unbounded this boundary contribution also vanishes under the condition that $\mathbf{P}_A$ and $\mathbf{P}_B$ have sufficient decay at infinity.
We conclude this subsection by analyzing reciprocity theorem (6) for some special cases.
Unbounded media – Consider the situation in which the medium at and outside $\partial\mathcal{V}_0$ is homogeneous, unbounded and source-free in both states. Assume that the wave fields in both states are causally related to the sources in $\mathcal{V}$. Then in both states the wave fields are outgoing at $\partial\mathcal{V}_0$ (i.e., $P_A^+ = P_B^+ = 0$ at the upper surface and $P_A^- = P_B^- = 0$ at the lower surface) and it is easily seen that $\mathbf{P}_A^T \underset{\sim}{\mathbf{N}} \mathbf{P}_B = P_A^+ P_B^- - P_A^- P_B^+ = 0$ at $\partial\mathcal{V}_0$, so the boundary integral on the left-hand side of equation (6) vanishes. Apparently it is not required that the medium parameters at and outside $\partial\mathcal{V}_0$ are identical in both states, unlike the conditions for the vanishing of the boundary integral in reciprocity theorem (2).

Physical reciprocity – Assume that the above mentioned conditions are fulfilled and that $\varrho_A = \varrho_B$ and $\kappa_A = \kappa_B$ inside as well as outside $\mathcal{V}$. Then the first volume integral on the right-hand side of equation (6) vanishes. Furthermore, consider point sources in states A and B at $\mathbf{x}_A \in \mathcal{V}$ and $\mathbf{x}_B \in \mathcal{V}$, respectively, according to $\mathbf{S}_A(\mathbf{x},\omega) = \mathbf{s}_A(\omega)\delta(\mathbf{x}-\mathbf{x}_A)$ and $\mathbf{S}_B(\mathbf{x},\omega) = \mathbf{s}_B(\omega)\delta(\mathbf{x}-\mathbf{x}_B)$. Equation (6) thus yields

$$\mathbf{P}_A^T(\mathbf{x}_B|\mathbf{x}_A,\omega)\underset{\sim}{\mathbf{N}}\mathbf{s}_B(\omega) = -\mathbf{s}_A^T(\omega)\underset{\sim}{\mathbf{N}}\mathbf{P}_B(\mathbf{x}_A|\mathbf{x}_B,\omega). \tag{8}$$

For the special case that $\mathbf{s}_A = (s_A^+\ 0)^T$ and $\mathbf{s}_B = (s_B^+\ 0)^T$, with $s_A^+ = s_B^+$, this reduces to (see Figure 4).

$$P_A^-(\mathbf{x}_B|\mathbf{x}_A,\omega) = P_B^-(\mathbf{x}_A|\mathbf{x}_B,\omega), \tag{9}$$

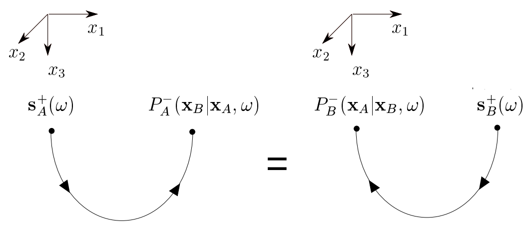

Figure 4: Physical reciprocity for one-way sources and receivers.

Reciprocity theorems for time-lapse seismics

Since in a reciprocity theorem two states interact, it is optimally fitted to formulate the relation between two measurements in a time-lapse seismic experiment. State A is associated with the reference wave field at, say, $t = t_1$, while state B is associated with the monitoring wave field at, say, $t = t_2 > t_1$. It is noted that $t_2 - t_1$ is much longer than the seismic experiment time. In our analysis $\mathbb{R}^3$ is divided in three domains (Figure 5): $\mathcal{V}_0$ is the domain where there are no differences between the material parameters in the two states, mostly associated with the domain above the reservoir; the domain $\mathcal{V}_c$, for example associated with the reservoir, where there is a difference between the material parameters in the two states mostly due to the reservoir production history; and $\mathcal{V}'$ denotes the complement of $\mathcal{V}_0 \cup \mathcal{V}_c$; the material parameters in this domain may or may not be different; a possible difference in this domain is taken into account in a subsequent step. The domains are specified as follows

$$\mathcal{V}_0 = \{\mathbf{x} \in \mathbb{R}^3, x_3 \le x_3^1\}, \mathcal{V}_c = \{\mathbf{x} \in \mathbb{R}^3, x_3^1 < x_3 \le x_3^2\} \quad \text{and} \quad \mathcal{V}' = \{\mathbf{x} \in \mathbb{R}^3, x_3 > x_3^2\}. \tag{10}$$

In the next subsections we will discuss the matter for the two reciprocity theorems discussed above.

In order to simplify the analysis we only consider point sources of the volume injection type. The source of state A is taken at $\mathbf{x} = \mathbf{x}_S$, while the source of state B is taken at $\mathbf{x} = \mathbf{x}_R$, according to

$$Q_A(\mathbf{x},\omega) = q_A(\omega)\delta(\mathbf{x}-\mathbf{x}_S) \quad \text{and} \quad Q_B(\mathbf{x},\omega) = q_B(\omega)\delta(\mathbf{x}-\mathbf{x}_R). \tag{11}$$

Application of reciprocity theorem (2) to domain $\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_c$ yields

$$\begin{aligned}
&\int_{x_3=x_3^2} \{P_A(\mathbf{x}|\mathbf{x}_S)V_{3,B}(\mathbf{x}|\mathbf{x}_R) - V_{3,A}(\mathbf{x}|\mathbf{x}_S)P_B(\mathbf{x}|\mathbf{x}_R)\}\mathrm{dA} \\
&= -j\omega \int_{\mathbf{x}\in\mathcal{V}_c} \{P_A(\mathbf{x}|\mathbf{x}_S)(\kappa_B(\mathbf{x}) - \kappa_A(\mathbf{x}))P_B(\mathbf{x}|\mathbf{x}_R) \\
&\qquad -V_{k,A}(\mathbf{x}|\mathbf{x}_S)(\varrho_B(\mathbf{x}) - \varrho_A(\mathbf{x}))V_{k,B}(\mathbf{x}|\mathbf{x}_R)\}\mathrm{dV} \\
&+q_B(\omega)P_A(\mathbf{x}_R|\mathbf{x}_S) - q_A(\omega)P_B(\mathbf{x}_S|\mathbf{x}_R).
\end{aligned} \tag{12}$$

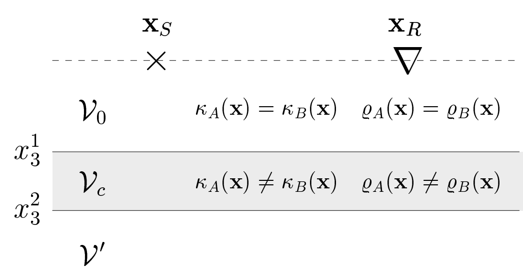

Figure 5: Configuration for time-lapse seismics.

Using physical reciprocity (equation 3) we arrive at

$$\begin{aligned}
&q_B(\omega)P_A(\mathbf{x}_R|\mathbf{x}_S) - q_A(\omega)P_B(\mathbf{x}_R|\mathbf{x}_S) \\
&\quad = j\omega \int_{\mathbf{x}\in\mathcal{V}_c} \{P_A(\mathbf{x}|\mathbf{x}_S)(\kappa_B(\mathbf{x}) - \kappa_A(\mathbf{x}))P_B(\mathbf{x}|\mathbf{x}_R) - V_{k,A}(\mathbf{x}|\mathbf{x}_S)(\varrho_B(\mathbf{x}) - \varrho_A(\mathbf{x}))V_{k,B}(\mathbf{x}|\mathbf{x}_R)\}\mathrm{dV} + \\
&\qquad \int_{x_3=x_3^2} \{P_A(\mathbf{x}|\mathbf{x}_S)V_{3,B}(\mathbf{x}|\mathbf{x}_R) - V_{3,A}(\mathbf{x}|\mathbf{x}_S)P_B(\mathbf{x}|\mathbf{x}_R)\}\mathrm{dA}.
\end{aligned} \tag{13}$$

The surface integral on the right-hand side of equation (13) takes into account a possible difference of the material parameters in $\mathcal{V}'$, below the reservoir; it vanishes when there is no difference between the two states in $\mathcal{V}'$.

In the one-way analysis we consider point-sources for downgoing waves in both states, according to

$$\mathbf{S}_A(\mathbf{x},\omega) = (s_A^+(\omega)\ 0)^T\delta(\mathbf{x}-\mathbf{x}_S) \quad \text{and} \quad \mathbf{S}_B(\mathbf{x},\omega) = (s_B^+(\omega)\ 0)^T\delta(\mathbf{x}-\mathbf{x}_R). \tag{14}$$

Application of reciprocity theorem (6) to domain $\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_c$ yields

$$\begin{aligned}
&\int_{x_3=x_3^2} \{P_A^+(\mathbf{x}|\mathbf{x}_S)P_B^-(\mathbf{x}|\mathbf{x}_R) - P_A^-(\mathbf{x}|\mathbf{x}_S)P_B^+(\mathbf{x}|\mathbf{x}_R)\}\mathrm{dA} \\
&\quad = \int_{\mathbf{x}\in\mathcal{V}_c} \mathbf{P}_A^T(\mathbf{x}|\mathbf{x}_S)\underset{\sim}{\mathbf{N}}(\hat{\underset{\sim}{\mathbf{B}}}_B(\mathbf{x}) - \hat{\underset{\sim}{\mathbf{B}}}_A(\mathbf{x}))\mathbf{P}_B(\mathbf{x}|\mathbf{x}_R)\mathrm{dV} - s_B^+(\omega)P_A^-(\mathbf{x}_R|\mathbf{x}_S) + s_A^+(\omega)P_B^-(\mathbf{x}_S|\mathbf{x}_R).
\end{aligned} \tag{15}$$

Using physical reciprocity (equation 9) we arrive at

$$s_B^+(\omega)P_A^-(\mathbf{x}_R|\mathbf{x}_S) - s_A^+(\omega)P_B^-(\mathbf{x}_R|\mathbf{x}_S) \tag{16}$$
$$= \int_{\mathbf{x}\in\mathcal{V}_c} \mathbf{P}_A^T(\mathbf{x}|\mathbf{x}_S)\underset{\sim}{\mathbf{N}}(\hat{\underset{\sim}{\mathbf{B}}}_B(\mathbf{x}) - \hat{\underset{\sim}{\mathbf{B}}}_A(\mathbf{x}))\mathbf{P}_B(\mathbf{x}|\mathbf{x}_R)\mathrm{dV} - \int_{x_3=x_3^2}\{P_A^+(\mathbf{x}|\mathbf{x}_S)P_B^-(\mathbf{x}|\mathbf{x}_R) - P_A^-(\mathbf{x}|\mathbf{x}_S)P_B^+(\mathbf{x}|\mathbf{x}_R)\}\mathrm{dA}.$$

As in the previous case, the surface integral on the right-hand side of equation (16) vanishes when there is no difference between the two states in $\mathcal{V}'$.

Examples

The material parameters for State A and State B, representing the monitor and the reference cases, respectively, are shown in Figure 6. For this specific case the boundary integral in equation (13) at different depth levels is calculated and we discuss the numerical results. Time-lapse changes are modeled inside the acquisition domain $V_{acq} \subset V_0$, $x_3 < 150\mathrm{m}$, and inside V_c, $250\mathrm{m} < x_3 < 700\mathrm{m}$, at $250\mathrm{m} < x_3 < 400\mathrm{m}$ and $500\mathrm{m} < x_3 < 700\mathrm{m}$. The interfaces are numbered from 1 to 5. Source and receivers are located at a depth of 10 m. In Figure 7 left, the difference of the monitor and reference scattered wave fields is shown. The difference reflection events are numbered according to the interfaces in Figure 6. Observe the phase differences between the reference and monitor reflections, which are top-down cumulative. The boundary integral of equation (13) is calculated at the bottom of the acquistion domain, see Figure 7 middle, and at the bottom of the first reservoir layer, see Figure 7 right, at 150 and 400 m, respectively. In Figure 7 middle the difference refection 1 has disappeared while difference reflection 2 is a pure amplitude difference without a phase difference. In Figure 7 right the difference reflections 1,2 and 3 have vanished while 4 is now a pure amplitude reflection. We can conclude that the boundary integral simulates the difference of two time-lapse experiments for which no temporal contrast exists above the particular surface over which we calculate the integral.

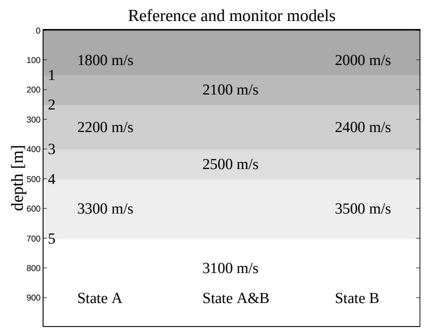

Figure 6: Velocities of State A (monitor state) and State B (reference state). Temporal changes occur in the first, third and fifth layer.

Discussion and conclusions

In this paper we have presented two formulations of the reciprocity theorem. Both theorems show how the acoustic states at the surface of some bounded domain are related to contrast functions and source distributions in this domain. In the reciprocity theorem for the full wave field the contrast function is expressed in material differences ($\Delta\kappa$, $\Delta\varrho$), while in the reciprocity theorem for the one-way wave fields it is expressed in scattering operators ($\hat{R}$, $\hat{T}$).
The reciprocity theorem for the full wave field has proven its functionality for example in the removal of multiple reflections (van Borselen *et al.*, 1996) and in velocity replacement (Smit *et al.*, 1998). The reciprocity theorem for the one-way wave fields has been the point of departure for the derivation of seismic imaging techniques for finely layered media (Wapenaar, 1996). As we have shown both reciprocity theorems are useful for time-lapse seismic imaging and inversion (Fokkema *et al.*, 1997).

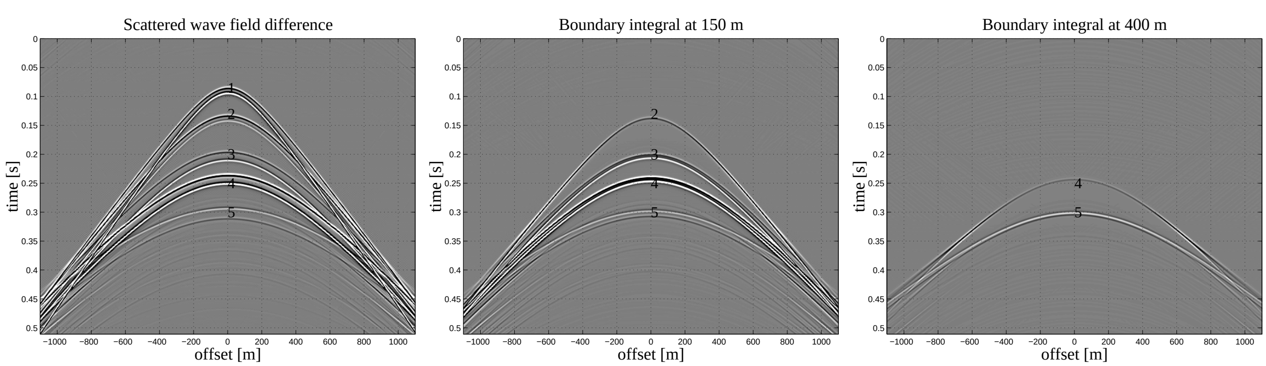

Figure 7: The difference of the monitor and the reference scattered wave fields with temporal changes identifiable as phase shifts in difference reflections (*left*). The boundary integral of equation (13) at 150 m (*middle*) and 400 m (*right*). Equivalent to a difference wave field without temporal contrasts above 150 m (*middle*) and 400m (*right*)