On the calculation of out-of-plane geometrical spreading in anisotropic media

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Abstract

The objective is to extend the generality of previously published formulae for the integrand inherent in the calculation of out-of-plane relative geometrical spreading. I obtain an integrand applicable to rays propagated in the symmetry plane of a monoclinic anisotropic medium. Integrands corresponding to "unrotated" orthorhombic and transversely isotropic media are derived as special cases.

Introduction

Two-dimensional (2D) models of the subsurface are frequently used for seismic ray modelling, especially in cases where the variations of model parameters in the direction normal to the model plane are negligible. The assumption of two-dimensionality is beneficial with respect to efficiency of computations and simplifies the ray tracing considerably. Even for a 2D model, however, a realistic simulation of wave propagation should be three-dimensional (3D), since, the wave amplitude will be influenced by in-plane as well as out-of-plane effects. The out-of-plane geometrical spreading can be obtained by including an additional integral equation in the dynamic ray tracing system. Out-of-plane dynamic ray tracing for anisotropic media has been described by Mispel (2001) for a transversely isotropic medium with a vertical axis of symmetry (i.e., a VTI medium) and by Ettrich et al. (2002) for an orthorhombic medium with symmetry planes coinciding with the three main planes of the model coordinate system ("unrotated" orthorhombic medium).

The objective of this paper is to extend the generality of previously published formulae for the integrand in out-of-plane dynamic ray tracing. I present a formula pertaining to the most general of the 2D anisotropic media, namely, the monoclinic medium. I also show that the new formula is consistent with previously presented forms of the integrand. A more complete description of ray tracing in a 2D anisotropic medium can be found in Iversen (2003b).

The context

I consider rays propagating in a 3D monoclinic anisotropic medium, specified with respect to a Cartesian model coordinate system with coordinates (x_1, x_2, x_3). The vertical plane $x_2 = 0$ is assumed to be a plane of mirror symmetry, and the paper is particularly concentrated on the special situation when rays are confined to this symmetry plane ("in-plane" propagation). The slowness vector and group velocity vector in a ray point $\mathbf{x} = (x_1, x_2, x_3)^T$ are denoted, respectively, $\mathbf{p} = (p_1, p_2, p_3)^T$ and $\mathbf{v} = (v_1, v_2, v_3)^T$. The components p_2 and v_2 are zero in all ray points. The start and end point of the ray are denoted by symbols S and R.

The density-normalized elastic coefficients are represented here in the Voigt notation. For the monoclinic medium with a symmetry plane $x_2 = 0$, the matrix of elastic coefficients has the form

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & A_{15} & 0 \\ . & A_{22} & A_{23} & 0 & A_{25} & 0 \\ . & . & A_{33} & 0 & A_{35} & 0 \\ . & . & . & A_{44} & 0. & A_{46} \\ . & . & . & . & A_{55} & 0 \\ . & . & . & . & . & A_{66} \end{bmatrix}. \qquad (1)$$

Furthermore, the geometric properties of rays in any anisotropic medium are determined by the characteristic equation

$$det(\Gamma - G\mathbf{I}) = \mathbf{0}\,. \qquad (2)$$

Here Γ is the Christoffel matrix (3 x 3), $\mathbf{I}$ is the identity matrix (3 x 3), and G is the eigenvalue corresponding to the wave under consideration. For ray propagation confined to a plane, there are three possible waves; the *P*- and *SV*-waves with in-plane polarization vectors, and the *SH*-wave with polarization vector directed normal to the plane. The matrix $\Gamma - G\mathbf{I}$ has a corresponding cofactor matrix,

$$\mathbf{D} \equiv cof(\Gamma - G\mathbf{I})\,, \qquad (3)$$

which is essential in the theory of rays in anisotropic media.

Out-of-plane geometrical spreading

Following Cerveny (2001, 359), the relative geometrical spreading due to a point source at S is

$$\mathcal{L}(R,S) = \left| det\,\mathbf{Q}^{(y)}(R,S) \right|^{1/2}. \qquad (4)$$

The 2 x 2 matrix $\mathbf{Q}^{(y)}(R,S)$ is given with respect to so-called wavefront-centered Cartesian coordinates (wavefront orthonormal coordinates), for which the third coordinate axis is perpendicular to the wavefront at R. Since $\mathbf{Q}^{(y)}(R,S)$ corresponds to point source initialization, we have $\mathbf{Q}^{(y)}(S,S) = \mathbf{0}$. If the ray is confined to the plane $x_2 = 0$ and the slowness vector component p_2 is zero, the matrix $\mathbf{Q}^{(y)}(R,S)$ will be a diagonal matrix. Therefore, the relative geometrical spreading factorizes,

$$\mathcal{L}(R,S) = \mathcal{L}^{||}(R,S)\mathcal{L}^{\perp}(R,S)\,, \qquad (5)$$

where the in-plane and out-of-plane geometrical spreadings are defined, respectively, by

$$\mathcal{L}^{||}(R,S) = \left| Q_{11}^{(y)}(R,S) \right|^{1/2},\ \mathcal{L}^{\perp}(R,S) = \left| Q_{22}^{(y)}(R,S) \right|^{1/2}. \qquad (6)$$

The matrix element elements $Q_{11}^{(y)}$ and $Q_{22}^{(y)}$ can be obtained, respectively, from in-plane dynamic ray tracing and from a closed-form integral (see the next section) For simplicity, the superscript $^{(y)}$ is dropped in the following.

Out-of-plane dynamic ray tracing

For point source initialization at the point *S*, the out-of-plane dynamic ray tracing system can be defined as

$$\frac{dQ_{22}}{dt} = T_{22} , \qquad \frac{dP_{22}}{dt} = 0 , \qquad (7)$$

where *t* is the traveltime along the ray. The equations (7) show that the quantity P_{22} is constant (=1) along the ray, with the consequence that the matrix element Q_{22} can be obtained by the integral

$$Q_{22}(R,S) = \int_{t(S)}^{t(R)} T_{22}(t)dt . \qquad (8)$$

Ettrich et al. (2002) showed that the integrand T_{22} is given by:

$$T_{22} = \lim_{p_2 \to 0} \frac{v_2}{p_2} . \qquad (9)$$

The latter expression is valid for an anisotropic medium with a symmetry plane x_2 = 0; in other words, media with monoclinic or higher-order symmetries. Ettrich et al. (2002) took expression (9) as a start point for further derivations. Here, however, I have used a different approach, which takes as a start point the general expression for the quantity T_{22} valid for 3D arbitrarily anisotropic and heterogeneous media (Cerveny, 2001; Iversen, 2003a). Thereby, I obtain the formula

$$T_{22} = \frac{1}{D}[D_{11}A_{66} + D_{22}A_{22} + D_{33}A_{44} + 2D_{13}A_{46} + (p_1, p_3)\mathbf{KLK}(p_1, p_3)^T] . \qquad (10)$$

The quantities D_{11}, D_{22}, D_{33}, and D_{13} are elements of the cofactor matrix **D** [equation (3)], and the denominator *D* is given by $D = D_{11} + D_{22} + D_{33}$. The quantities **K** and **L** are symmetric 2 x 2 matrices, with definitions

$$\mathbf{K} = \begin{bmatrix} A_{12} + A_{66} & A_{25} + A_{46} \\ A_{25} + A_{46} & A_{23} + A_{44} \end{bmatrix}, \mathbf{L} = \begin{bmatrix} G - \Gamma_{33} & \Gamma_{13} \\ \Gamma_{13} & G - \Gamma_{11} \end{bmatrix} . \quad (11)$$

The formula (10) is valid for *P*/*SV*-rays and *SH*-rays propagating in the symmetry plane x_2 = 0 (p_2 = 0) of a monoclinic medium. For a *P*/S*V*-ray we have D_{22} = 0 and $D = D_{11} + D_{33}$, For an *SH*-ray we have $D_{11} = D_{33} = D_{13} = 0$ and $D = D_{22}$.

Note that it is also possible to obtain the formula (10) on the basis of equation (9). Ettrich et al. (2002) presented a formula similar to (10), but the generality was limited to an unrotated orthorhombic medium. By setting $A_{15} = A_{35} = A_{46} = A_{25} = 0$, it is easily shown that my formula (10) is consistent with Ettrich et al.'s result.

For a VTI medium, one can show that the characteristic equation (2) factorizes solely due to the requirements on the medium parameters. Thus, in this case it is rather straightforward to obtain the integrand T_{22} by differentiating the relevant factor of the characteristic equation (corresponding either to *P*/*SV*- or *SH*-rays) twice with respect to the slowness component p_2. Here, however, I have derived the integrand T_{22} directly from the formula (10). The result for *P*/*SV*-rays is

$$T_{22} = \frac{G(A_{11} + A_{55}) - 2A_{11}A_{55}p_1^2 - lp_3^2}{2G - (A_{11} + A_{55})p_1^2 - (A_{33} + A_{55})p_3^2} , \qquad (12)$$

where

$$l \equiv A_{11}A_{33} + A_{55}^2 - (A_{13} + A_{55})^2 . \qquad (13)$$

Formula (12) is consistent with the derivation done by Mispel (2001). For an *SH*-ray in a VTI medium, I obtain simply

$$T_{22} = A_{66} . \qquad (14)$$

The formulae (10) and (12) include the eigenvalue *G*, which is 1 for a ray integrated without errors. For rays contaminated with small errors, recent tests (Mispel, 2001) indicate that insertion of the actual value of *G*, corresponding to the inaccurate ray, has a stabilizing effect on the ray propagation.

Conclusions

Ray theory for the calculation of the out-of-plane geometrical spreading, corresponding to a symmetry plane within a monoclinic anisotropic medium, is addressed in this paper. I provide a generalized expression for the integrand in the out-of-plane dynamic ray tracing. The formula is consistent with recently published formulae for media with less general anisotropy than the monoclinic medium [VTI medium (Mispel, 2001); unrotated orthorhombic medium (Ettrich et al., 2002)]. The formulae for the integrand pertaining to a monoclinic or an orthorhombic medium can not be used in the vicinity of shear-wave singularities. However, the presented integrands (12) and (14) for out-of-plane dynamic ray tracing in VTI media can be used without problems in such situations.

References

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