Impedance-type approximations of the P–P elastic reflection coefficient: Modeling and AVO inversion

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Abstract

The normal-incidence elastic compressional reflection coefficient admits an exact, simple expression in terms of the acoustic impedance, namely, the product of the P-wave velocity and density, at both sides of the interface. With slight modifications a similar expression can, also exactly, express the oblique-incidence acoustic reflection coefficient. A severe limitation on the use of the above two reflection coefficients in analyzing seismic reflection data is that they provide no information on shear-wave velocities that refer to the interface. In this paper, we address the natural question of whether a suitable impedance concept can be introduced for which arbitrary P-P reflection coefficients can be expressed in an analogous form as their counterpart acoustic ones. We formulate this problem by considering the mathematical conditions to be satisfied by such a general impedance function. Although no closed-form exact solution exists, our analysis provides a general framework for which, under suitable restrictions of the medium parameters, possible impedance functions can be derived. In particular, the well-established concept of elastic impedance and the recently introduced concept of reflection impedance can be better understood. Concerning these two impedances, we examine their potential for modelling and for the estimation of the AVO indicators of intercept and gradient. For typical synthetical examples, we show that the reflection impedance formulation provides consistently better results than those obtained using the elastic impedance.

Introduction

The compressional wave (P-P) reflection coefficient in acoustic media (S-wave velocity $\beta = 0$) is given by

$$R = \frac{AI_2\ \sec\theta_2 - AI_1\ \sec\theta_1}{AI_2\ \sec\theta_2 + AI_1\ \sec\theta_1}\,, \qquad AI_i = \rho_i\,\alpha_i\,, \tag{1}$$

where ρ_i and α_i denote the density and P-wave velocity, respectively, at the incident side $(i = 1)$ and at the opposite side $(i = 2)$ of the reflecting interface, AI is the *acoustic impedance* function, θ_1 is the incidence angle and θ_2 is the transmited angle, satisfing the Snell's law $\alpha_2\ \sin\theta_1 = \alpha_1\ \sin\theta_2$. For elastic media (S-wave velocity $\beta \neq 0$), the expression for the P-P reflection coefficient is also the ratio between two quantities,

$$R = \frac{P[\rho,\alpha,\beta,\theta]}{Q[\rho,\alpha,\beta,\theta]}\,. \tag{2}$$

However, the numerator (P) and denominator (Q) do not have the simple form as in the acoustic case (see, e.g., Aki & Richards (1980)).

As seen by the recent literature (see, e.g., Connolly (1999); Mallick (2001)), it makes sense to look for a quantity (impedance) $I \equiv I(\rho,\alpha,\beta,\theta)$ for which the P-P reflection coefficient can be given, at least approximately, by an expression of the form

$$R = \frac{I_2 - I_1}{I_2 + I_1}\,. \tag{3}$$

The reflectivity function

Roughly speaking, the reflectivity function is a measure of the variation of the reflection coefficient as we move along a ray within a layered media. To quantitatively express this variation, we consider that the elastic characteristics, ρ, α and β, as well as the incident angle, θ, are functions of a single variable, σ, that parameterize the ray. This variable can be, e.g., depth or time. In other words, we consider, along the ray, the vector quantity $\eta(\sigma) = (\rho(\sigma),\alpha(\sigma),\beta(\sigma),\theta(\sigma))$. With this understanding, we can recast the reflection coefficient, as given by equation (2), in the form

$$R \equiv R(\sigma,\Delta\sigma) = \frac{P[\eta(\sigma),\eta(\sigma+\Delta\sigma)]}{Q[\eta(\sigma),\eta(\sigma+\Delta\sigma)]}\,. \tag{4}$$

where $\Delta\sigma$ is the parameter increment, chosen to be sufficiently small. In the above formula σ and $\sigma+\Delta\sigma$ replace indices 1 and 2, respectively. For example, $\rho(\sigma)$ replaces ρ_1, $\alpha(\sigma+\Delta\sigma)$ replaces α_2, etc.

The P-P *elastic reflectivity* function $\mathcal{R}$ can be defined as the limit,

$$\mathcal{R}(\sigma) = \lim_{\Delta\sigma\to 0}\frac{R(\sigma,\Delta\sigma)}{\Delta\sigma}\,. \tag{5}$$

Using the *exact* formula for the P-P reflection coefficient (Aki & Richards, 1980), we readily obtain the expression

$$\mathcal{R}(\sigma) = \left[1 - 4\beta^2p^2\right]\frac{\rho'}{2\rho} + \left[\frac{1}{1-\alpha^2p^2}\right]\frac{\alpha'}{2\alpha} - \left[4\beta^2p^2\right]\frac{\beta'}{\beta}\,, \tag{6}$$

where the prime denotes derivative with respect to to σ and p is the ray parameter given by $p = \sin\theta/\alpha$. Under the assumption of a flat-layered medium, the ray parameter is assumed to be constant along the ray.

Using the reflectivity function definition (5), and approximating the derivatives in equation (6) by their corresponding discrete differences, i.e., $f' \approx \Delta f/\Delta\sigma$, we arrive at the well-known first-order approximation for R (Aki & Richards, 1980),

$$\begin{aligned} R \approx \mathcal{R}(\sigma)\,\Delta\sigma \quad &\approx \quad \frac{1}{2}\left[1 - 4\frac{\beta^2}{\alpha^2}\sin^2\theta\right]\frac{\Delta\rho}{\rho} \\ &+ \quad \frac{1}{2}\left[\sec^2\theta\right]\frac{\Delta\alpha}{\alpha} \\ &- \quad \left[4\frac{\beta^2}{\alpha^2}\sin^2\theta\right]\frac{\Delta\beta}{\beta}\ . \end{aligned} \tag{7}$$

For sufficiently small incidence angles, $\tan^2\theta \approx \sin^2\theta$, and then we may rewrite equation (7) as the well-known Intercept and Gradient formula given in Shuey (1985), namely

$$R \approx A + B\sin^2\theta\ , \tag{8}$$

with

$$A = \frac{1}{2}\left[\frac{\Delta\rho}{\rho} + \frac{\Delta\alpha}{\alpha}\right]\ , \tag{9}$$

and

$$B = \frac{1}{2}\frac{\Delta\alpha}{\alpha} - 2\frac{\beta^2}{\alpha^2}\left[\frac{\Delta\rho}{\rho} + 2\frac{\Delta\beta}{\beta}\right]\ . \tag{10}$$

The problem of finding a function I satisfying equation (3) *exactly* is equivalent to that of determining a solution of the differential equation resulting from the computation of the limit in (5), assuming the desired form (3):

$$\begin{gathered} \lim_{\Delta\sigma\to 0}\left[\frac{1}{\Delta\sigma}\ \frac{I(\sigma+\Delta\sigma) - I(\sigma)}{I(\sigma+\Delta\sigma) + I(\sigma)}\right] = \frac{1}{2}\frac{I'(\sigma)}{I(\sigma)} = \\ \frac{1}{2}\left[1 - 4\beta^2p^2\right]\frac{\rho'}{\rho} + \frac{1}{2}\left[\frac{1}{1-\alpha^2p^2}\right]\frac{\alpha'}{\alpha} - \left[4\beta^2p^2\right]\frac{\beta'}{\beta}\ . \end{gathered} \tag{11}$$

The Impedance Function

The *Elastic Impedance* function EI proposed by Connolly (1999) is obtained by equalling equation (7) to $\Delta I/2I$ (the discrete version of $I'/2I$) and applying difference calculus, with the additional assumption that θ and the ratio $K = \beta^2/\alpha^2$ are constant. The same result can be found directly from solving equation (11) under the mentioned assumptions:

$$I \equiv EI = M_0\ \rho^{1 - 4K\sin^2\theta}\ \alpha^{\sec^2\theta}\ \beta^{-8K\sin^2\theta}\ , \tag{12}$$

where M_0 is a normalization constant (Whitcombe, 2002).

The question now is if there is a *Reflection Impedance* function RI, solution of equation (11), for all possible choices of α, β and ρ. Clearly, the solution is not unique, since any multiple of it is also a solution. Equation (11) admits a closed-form solution only if β has a functional dependence on ρ, i.e., $\beta \equiv \beta(\rho)$. A particularly simple formula is obtained by assuming a relationship of the form $\rho = b\ \beta^\gamma$, or equivalently, $\rho'/\rho = \gamma\ \beta'/\beta$, where b is some constant of proportionality and γ is a constant. In this case, if $\beta' \neq 0$,

$$I \equiv RI = N_0\ \frac{\rho\ \alpha}{\sqrt{1-\alpha^2p^2}}\ \exp\{-2[2+\gamma]\beta^2p^2\}\ , \tag{13}$$

where N_0 is a constant.

In the case of a normal incidence, both elastic (EI) and the reflection (RI) impedance functions reduce to a multiple of the acoustic impedance (AI), so the approximation for the reflection coefficient remains exact. However, for the case of non-normal incidence in acoustic media ($\beta = 0$), the elastic impedance approximation for R *does not reduce* to the exact one given by equation (1), as opposite to the reflection impedance approximation, where the exact expression is maintained.

Applications

In order to analyse the accuracy of EI and RI functions presented above, we consider a simple two-layer model with large contrasts of the parameters:

Table 1: P- and S-wave velocities and densities

Medium	α [km/s]	β [km/s]	ρ [g/cm^3]
Layer 1	4.50	2.10	2.70
Layer 2	3.00	1.40	2.20
Contrast	0.40	0.40	0.20

We compare the exact reflection coefficient with its first-order approximation (see equation (7)), as well as the impedance-type approximations of equation (3) under the use of the elastic impedance of equation (12) and reflection impedance of equation (13), respectively. Observe that the ratio β/α in order to offer the best conditions for the elastic impedance approximation. The values for the constants M_0 and N_0 are irrelevant: any choice will produce the same value for the approximation of R. We also consider the two possibilities for the incidence layer: layer 1 (noncritical reflections) or layer 2 (post-critical reflections). The resulting approximations for the reflection coefficient is shown in Figure 1.

We have also compared the performance of the three different approximations of R for the estimation of the intercept, A, and gradient, B, attributes, according to equation (9). The model parameters are the same as in the previous experiments. We have added a white noise of ratio 1:3 to the exact reflection coefficient's curve and then apply least-squares techniques to recover A and B. The table below summarizes the inversion results, where, again, we can observe that the inverted attributes using the reflection impedance approximation are of better accuracy than all the others.

Table 2: Results of the least-squares inversion

	Incidence	Exact	Linear	EI	RI
A	$1 \to 2$	– 0.30	– 0.23	– 0.21	– 0.28
	$2 \to 1$	0.30	0.57	0.53	0.33
B	$1 \to 2$	0.24	– 0.18	– 0.31	0.12
	$2 \to 1$	– 0.24	– 1.40	– 0.88	– 0.47

Figure 2 shows the approximation for R using the inverted parameters and the corresponding approximation formulas.

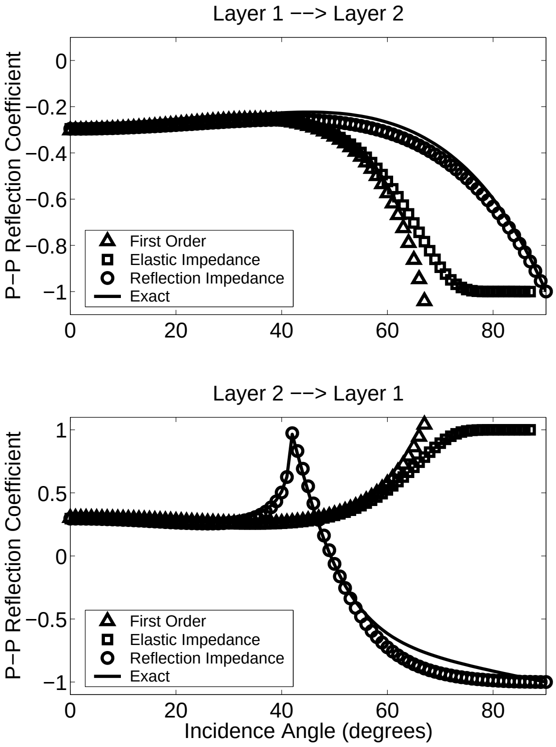

Figura 1: P-P reflection coefficient.

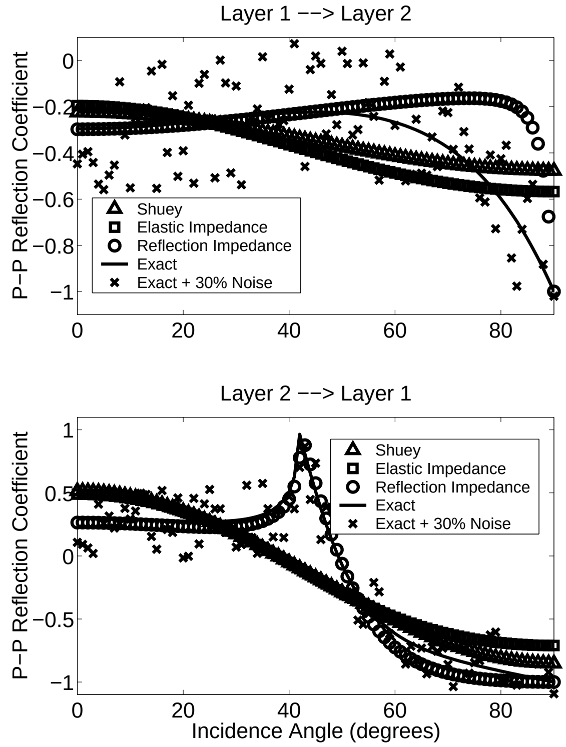

Figura 2: AVO curves

Conclusions

We have discussed the problem of determination and use of impedance functions generalizing the simple expression of the P-P reflection coefficient under normal incidence in acoustic/elastic media under oblique incidence in acoustic media, to oblique-incidence in elastic media. We have shown that for arbitrary selection of densities and P- and S-velocities, there is no closed-form impedance function fulfills the required task. Under additional, ad hoc, assumptions, impedance functions can be defined that provide useful approximations to the P-P reflection coefficients.

Our simple, but typical, numerical experiments have shown that the reflection impedance provide significantly better results, both for modelling and AVO inversion. The reflection impedance approximation has the best performance in all cases. In the case of post-critical reflections, the results are far better: all other approximations do not follow the correct shape of the exact curve. Therefore, there is a significant gain in accuracy provided by the reflection impedance approximation, as compared to the one that uses the elastic impedance.

Current research is being done to employ a similar approach using, however the reflection impedance function. First results in this direction are shown in Santos et al. (2002).

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