Born, Kirchhoff, and Born–Kirchhoff modeling: a comparison

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Abstract

For the modeling of a single target reflector in a smooth inhomogeneous elastic anisotropic media, the volume Born integral can be transformed into a surface scattering integral on the reflector. This surface integral, called Born-Kirchhoff integral, relates very naturally to the Kirchhoff-Helmholtz integral, thus providing the theoretical link between the two approaches. Here we specialize the derivation and main properties of the Born-Kirchhoff integral in the acoustic case, and use simple synthetic examples to provide a comparison between the new integral and its classical counterparts.

Introduction.

The Born (volume) and Kirchhoff-Helmholtz (surface) representation integrals are the most widely used descriptions of reflected and transmitted wavefields due to smooth interfaces (see, e.g., Bleistein, 1984; Wapenaar and Berkhout, 1993; Chapman and Coates, 1994; Tygel et al., 1994; and Schleicher, et al., 2001).

Although representing basically the same phenomena, the two integrals result from quite independent formulations, and are traditionally kept as completely separate objects. Moreover, besides their fundamental distinction as volume and surface integrals (Wapenaar and Berkhout, 1993), the representations of Born and Kirchhoff-Helmholtz present also other differences, namely (a) Born assumes weak medium perturbations, uses a linearized scattering coefficient and the resulting integral is reciprocal and (b) Kirchhoff-Helmholtz imposes no contrast restrictions for the medium inhomogeneities, approximates the reflected field and its normal derivative on the reflector using the plane-wave reflection coefficient and the incident field, and the resulting integral is nonreciprocal. When evaluated by means of the stationary phase method, the Kirchhoff-Helmholtz integral yields the ray-theoretical expressions (Schleicher et al., 2001).

The Born volume integral can be transformed into a corresponding surface scattering integral by application of a generalized form of the divergence theorem (Ursin and Tygel, 1997; Novais et al., 1997). This new integral, called the Born-Kirchhoff integral by Ursin and Tygel (1997) in the context of elastic, anisotropic media, provides the natural theoretical link between the Born and Kirchhoff-Helmholtz representations.

In this work, we provide a quick derivation of the Born-Kirchhoff integral for the case of acoustic inhomogeneous media, as well as summarize its main properties. Furthermore, we examine its application to simple synthetic examples to compare the obtained results with the ones corresponding to its classical counterparts. We also take into the comparison a modified, reciprocal, Kirchhoff-Helmholtz integral introduced by Deregowski and Brown (1983).

Formulation of the problem.

We consider a scattering medium consisting of two unbounded, inhomogeneous acoustic halfspaces, separated by a smooth interface Σ. We also consider a reference medium characterized by smooth compression modulus $k(\mathbf{x})$ and smooth mass density $\rho(\mathbf{x})$, where $\mathbf{x} = (x, y, z)$ denotes the location vector in a fixed, global Cartesian coordinate system. The model parameters of the upper halfspace coincide with those of the reference medium. The lower halfspace has perturbed parameters $k(\mathbf{x}) + \Delta k(\mathbf{x})$ and $\rho(\mathbf{x})+\Delta\rho(\mathbf{x})$. The total acoustic pressure due to a point source located at $\mathbf{x}^s = (x^s, y^s, z^s)$ in the upper halfspace, is denoted in the frequency domain by $P = P(\mathbf{x}, \omega; \mathbf{x}^s)$. It satisfies the acoustic Helmholtz equation

$$\nabla \cdot \left[\frac{1}{\tilde{\rho}(\mathbf{x})}\nabla P\right] + \frac{\omega^2}{\tilde{k}(\mathbf{x})}P = -F(\omega)\delta(\mathbf{x}-\mathbf{x}^s) \ , \qquad (1)$$

where $F(\omega)$ is the source function and ω is the angular frequency. Moreover, $\tilde{\rho}$ and $\tilde{k}$ denote the acoustic parameters for the scattering medium in the upper halfspace

$$\tilde{\rho}(\mathbf{x}) = \rho(\mathbf{x}) \ , \tilde{k}(\mathbf{x}) = k(\mathbf{x}) \ , \qquad (2)$$

and in the lower halfspace

$$\tilde{\rho}(\mathbf{x}) = \rho(\mathbf{x}) + \Delta\rho(\mathbf{x}) \ , \tilde{k}(\mathbf{x}) = k(\mathbf{x}) + \Delta k(\mathbf{x}) \ . \qquad (3)$$

For observation points in the upper halfspace, the total pressure field can be decomposed into the superposition $P(\mathbf{x}, \omega; \mathbf{x}^s) = P^I(\mathbf{x}, \omega; \mathbf{x}^s) + P^R(\mathbf{x}, \omega; \mathbf{x}^s)$, where P^I is the *incident wavefield*, $P^I(\mathbf{x}, \omega; \mathbf{x}^s) = F(\omega)\ G^s(\mathbf{x}, \omega)$, and P^R is the *scattered* or *reflected* wavefield. Here, $G^s(\mathbf{x}, \omega)$ is the Green's function in the reference medium (which coincides with the scattering medium in the upper halfspace) for a point source at $\mathbf{x}^s$, observed at $\mathbf{x}$.

Born approximation.

Under the assumption of low contrasts $|\Delta\rho/\rho| \ll 1$ and $|\Delta k/k| \ll 1$, as well as of weak scattering $|P^R| \ll |P^I|$ and $|\nabla P^R| \ll |\nabla P^I|$, the scattered field can be approximated by the familiar Born representation (see, e.g., Wapenaar and Berkhout, 1993),

$$P^R(\mathbf{x}^r, \omega; \mathbf{x}^s) \approx P^{\mathcal{B}}(\mathbf{x}^r, \omega; \mathbf{x}^s) = F(\omega) \int_\Omega dV \left\{ \frac{\Delta\rho}{\rho^2}\nabla G^s \cdot \nabla G^r - \omega^2\ \frac{\Delta k}{k^2} G^s\ G^r \right\} \ . \qquad (4)$$

Here, the domain of integration Ω is the region of nonzero perturbations, namely the lower halfspace, and G^r is Green's function due to a point source placed at the receiver location. As easily verified, the resulting approximation $P^{\mathcal{B}}$ is reciprocal.

Kirchhoff-Helmholtz approximation.

The Kirchhoff-Helmholtz approximation of the scattered field is based on the assumption that at the interface Σ, the scattered field and its normal derivative are well-approximated by the incident field multiplied by the plane-wave reflection coefficient $R(\theta^s)$ (see, e.g., Wapenaar and Berkhout, 1993), i.e.,

$$P^R \approx R(\theta^s)\ P^I \ , \quad \frac{\partial P^R}{\partial \eta} \approx -\ R(\theta^s)\ \frac{\partial P^I}{\partial \eta} \ , \tag{5}$$

where θ^s is the incidence angle. The Kirchhoff-Helmholtz surface integral approximation reads (see, e.g., Tygel et al., 1994)

$$P^R(\mathbf{x}^r, \omega; \mathbf{x}^s) \approx P^{\mathcal{KH}}(\mathbf{x}^r, \omega; \mathbf{x}^s) = F(\omega) \int_\Sigma dS\ \frac{R(\theta^s)}{\rho} \left\{ G^r\ \frac{\partial G^s}{\partial \eta} + G^s\ \frac{\partial G^r}{\partial \eta} \right\} \ . \tag{6}$$

Note that $R(\theta^s)$ depends on the reflector geometry and the resulting approximation $P^{\mathcal{KH}}$ is nonreciprocal.

High-frequency ray approximation.

We now use the high-frequency, zero-order, ray approximation of the Green's function G and its gradient ∇G,

$$G^j(\mathbf{x}, \omega) = A^j(\mathbf{x})\ \exp\left\{ i\omega T^j(\mathbf{x}) \right\}, \tag{7}$$

$$\nabla G^j(\mathbf{x}, \omega) = i\omega\ \nabla T^j(\mathbf{x})\ G^j(\mathbf{x}, \omega), \tag{8}$$

where $A^j(\mathbf{x}) = A(\mathbf{x}; \mathbf{x}^j)$ is the amplitude factor and $T^j = T(\mathbf{x}, \mathbf{x}^j)$ is the traveltime along the ray segment $\mathbf{x}$ and $\mathbf{x}^j$, in which $j = s$ (source point) or $j = r$ (receiver). The use of these expression in the Born and Kirchhoff-Helmholtz representation integrals (4) and (6) yields

$$P^{\mathcal{B}}(\mathbf{x}^r, \omega; \mathbf{x}^s) = -\ \omega^2 F(\omega) \int_\Omega dV\ \frac{R_L(\theta^{rs}/2)}{\rho} \left[\frac{4\cos^2(\theta^{rs}/2)}{v^2} \right]\ G^s\ G^r \tag{9}$$

and

$$P^{\mathcal{KH}}(\mathbf{x}^r, \omega; \mathbf{x}^s) = i\omega F(\omega) \int_\Sigma dS\ \frac{R(\theta^s)}{\rho}\ \left(\frac{\cos\theta^s + \cos\theta^r}{v} \right)\ G^s\ G^r \ , \tag{10}$$

where

$$R_L(\theta^{rs}/2) = \left[\frac{\Delta k}{k} + \frac{\Delta \rho}{\rho} \cos\theta^{rs} \right] \frac{1}{4\cos^2(\theta^{rs}/2)} \ . \tag{11}$$

Here $\theta^{rs} = \theta^s + \theta^r$, where θ^s and θ^r are the incidence angles of the source and receiver rays at $\mathbf{x}$.

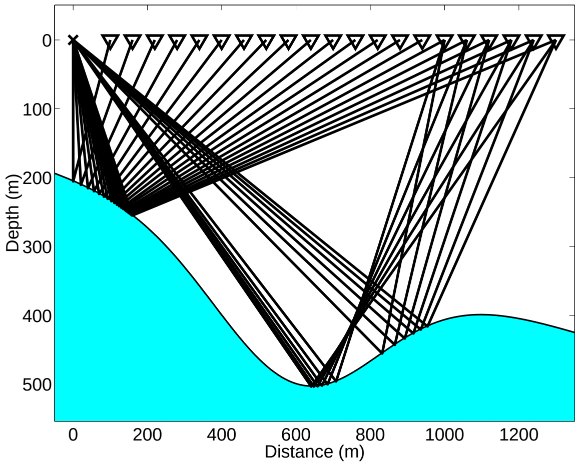

Figure 1: Model.

Born-Kirchhoff integral.

The above volume Born integral (9) can be transformed, still within the high-frequency ray approximation, into a surface integral which is very similar to the ray Kirchhoff-Helmholtz integral (10). Application of the high-frequency form of the divergence theorem to the ray Born integral (9) results in the Kirchhoff-Born integral (see Tygel and Ursin, 1997)

$$P^{\mathcal{BK}}(\mathbf{x}^r, \omega; \mathbf{x}^s) \approx i\omega F(\omega) \int_\Sigma dS\ \frac{R_L(\theta^{rs}/2)}{\rho}\ \left(\frac{\cos\theta^s + \cos\theta^r}{v} \right)\ G^s\ G^r \ . \tag{12}$$

A simple comparison between the Born-Kirchhoff integral (12) with its classical Kirchhoff-Helmholtz counterpart (10) shows that they only differ by the reflection coefficient and incident angle employed. For each point on the reflector Σ, the Born-Kirchhoff integral utilizes the weak-contrast, linearized, plane-wave reflection coefficient $R_L(\theta^{rs}/2)$ computed for half the angle θ^{rs} between the two ray segments that join this point to the fixed source–receiver pair. For the same point on the reflector, the Kirchhoff-Helmholtz integral uses the full plane-wave reflection coefficient $R(\theta^s)$ and the angle θ^s between the source ray segment and the surface normal. It is clear that both angles are equal when the surface point is a specular reflection point. It is to be noted that Deregowski and Brown (1983), concerned with the non-reciprocity of the the classical Kirchhoff-Helmholtz integral, have heuristically proposed the introduction of half total source-receiver angle θ^{rs} instead of the incident angle θ^s in the computation of the plane-wave reflection coefficient. We call this approximation Reciprocal Kirchhoff-Helmholtz. Our results show that, at least under the weak-scattering condition, the Reciprocal Kirchhoff-Helmholtz approximation can be well justified.

Numerical experiments.

To test the four different integral representations discussed above, we have modeled synthetic seismic section for a

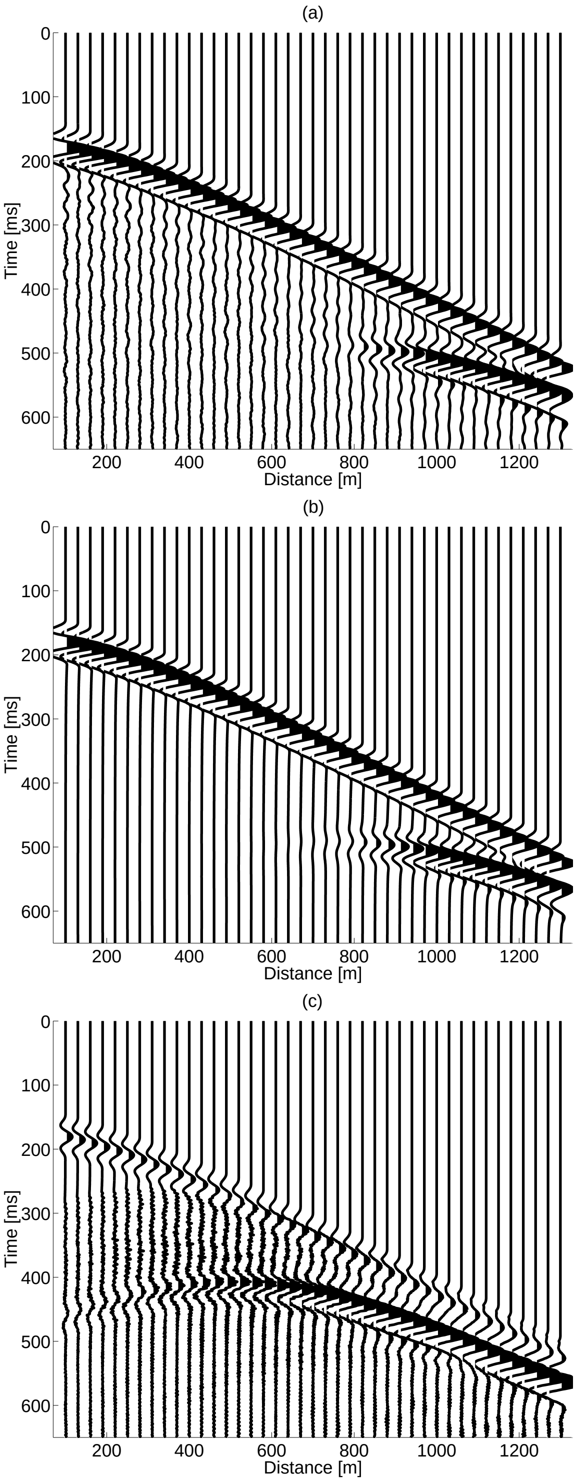

Figure 2: (a) Finite-difference section, (b) Born-Kirchhoff section, (c) Kirchhoff-Helmholtz section.

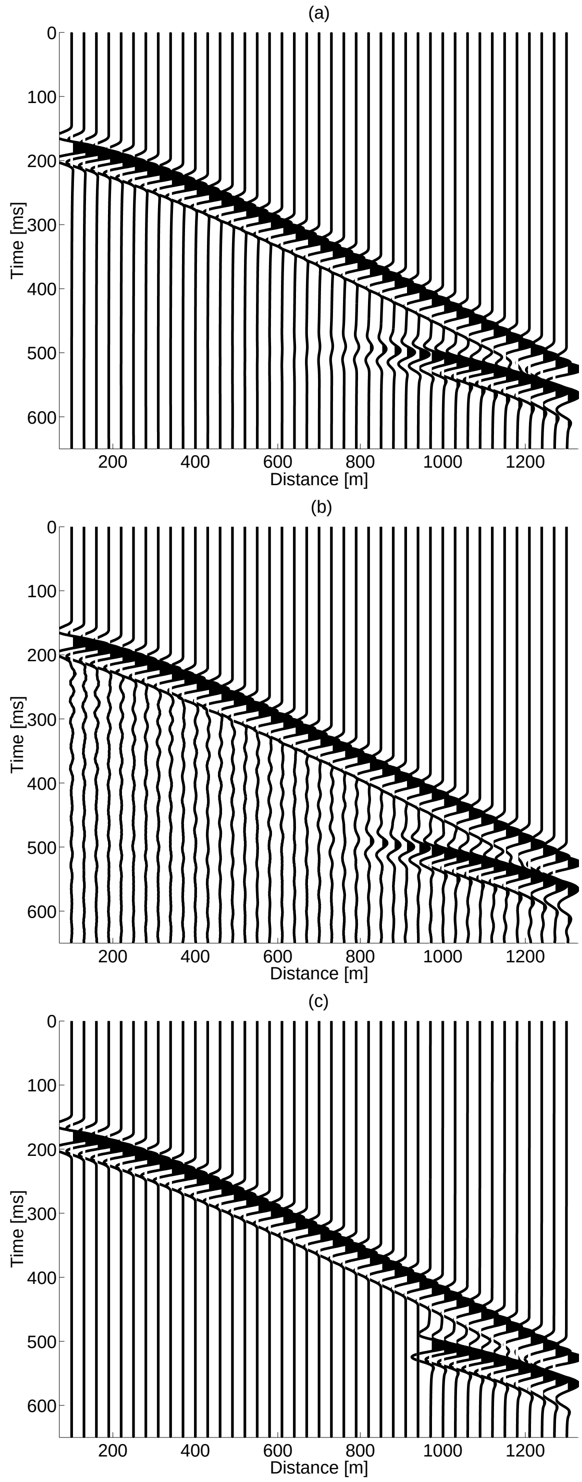

Figure 3: (a) Reciprocal Kirchhoff-Helmholtz section, (b) Born section, (c) Ray section.

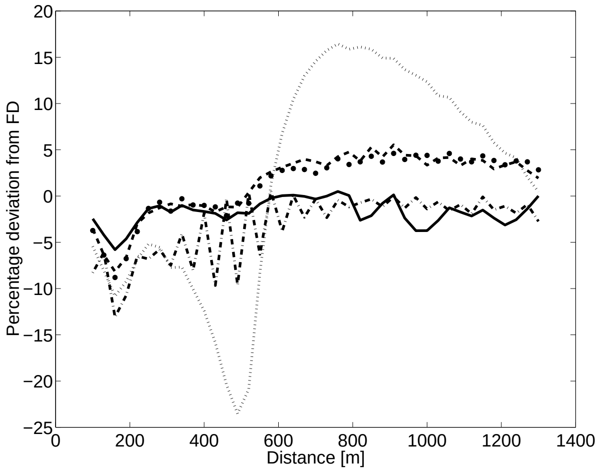

Figure 4: Percentage deviation of the amplitudes along the first reflection event from the finite-differences result. Born-Kirchhoff: solid line (—); Ray: dashed line (- -); Born: dash-dotted line (-·-); classical Kirchhoff-Helmholtz: small dots (···); Reciprocal Kirchhoff: large dots (• • •).

number of earth models. The results for a common-shot experiment from one of these models are depicted in Figures 2 and 3. Figure 1 shows the earth model consisting of two homogeneous halfspaces with velocities 3.0 km/s above and 3.5 km/s below a curved interface. The density in the medium is constant and equal to unity.

Figures 2 and 3 show the synthetic common-offset sections as obtained from finite-differences (second-order in time and fourth-order in space) (Figure 2a), the Born-Kirchhoff (Figure 2b), Kirchhoff-Helmholtz (Figure 2c), Reciprocal Kirchhoff-Helmholtz (Figure 3a) and Born (Figure 3b) representations, as well as from ray modeling (Figure 3c). To compute the integrals in all cases we used the trapezoidal rule with a uniform spatial grid $\Delta x = \Delta y = \Delta z = 10$ m and a time sampling of $\Delta t = 1$ ms. For the point source we have chosen a Küpper wavelet with a length of 72 ms.

Several differences between the modeling results can already be noted in the synthetic sections. For a more quantitative analysis, we have picked the peak amplitude along the first arrival. Figure 4 shows their deviations from the FD result. Observe that the Born-Kirchhoff integral exhibits the smallest error over the whole range of offsets.

Conclusions.

We have numerically investigated different integral approximations to the reflected acoustic wavefield. In all our numerical experiments, all above integrals approximate the reflected wavefield (as calculated by the Finite Difference method) quite well. However, the quality of the approximation depends on the investigated model. A method that would generally provide the best approximation cannot be determined. In spite of that, the Born-Kirchhoff integral proved to be the most stable approximation that yielded in all studied examples a result of good quality, either the best one or close to the best one. Moreover, the computation time for this method is (together with that for the Reciprocal Kirchhoff approximation) the smallest one of all methods under investigation.

Acknowledgments.

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References

[1] Bleistein, N., 1984, Mathematical Methods for Wave Phenomena. Academic Press Inc..

[2] Chapman, C. H., and Coates, R.V., 1994, Generalized Born Scattering in Anisotropic Media. Wave Motion, 19: 309-341.

[3] Deregowski, S. M., and Brown, S. M., 1983, A Theory of Acoustic Diffractors Applied to 2-D Models. Geophysical Prospecting, 31: 293–333.

[4] Tygel, M., Schleicher, J., and Hubral, P., 1994, Kirchhoff–Helmholtz Theory in Modeling and Migration. Journal of Seismic Exploration, 3: 203–214.

[5] Novais, A., Santos, L.T., Tygel, M., and Ursin, B., 1997, A Unified Born–Kirchhoff Representation for Acoustic Media. Journal of Seismic Exploration, 6: 217–226.

[6] Schleicher, J., Tygel, M., Ursin, B., and Bleistein, N., 2001, The Kirchhoff-Helmholtz integral for anisotropic elastic media, Wave Motion, 34: 353-364.

[7] Ursin, B., and Tygel, M., 1997, Reciprocal Volume and Surface Scattering Integrals for Anisotropic Elastic Media. Wave Motion, 26: 31 – 42.

[8] Wapenaar, C. P. A., and Berkhout, A.J., 1993, Representations of Seismic Reflection Data. Journal of Seismic Exploration, 2: 123–131.