Theory of True Amplitude One-way Wave Equations and True Amplitude Common-shot Migration

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Abstract

One-way wave operators are powerful tools for forward modeling and inversion. Their implementation, however, involves introducing the square-root of an operator as a pseudo-differential operator. A simple factoring of the wave operator produces one-way wave equations that yield the same traveltimes as the full wave equation, but do not yield accurate amplitudes except for homogeneous media and for some rare points in heterogeneous media. Here, we present augmented one-way wave equations. We show that these equations yield solutions for which the leading order asymptotic amplitude as well as the traveltime satisfy the same differential equations as do the corresponding functions for the full wave equation. Exact representations of the square-root operator appearing in these differential equations are elusive, except in cases in which the heterogeneity of the medium is independent of the transverse-spatial variables. Here, singling out depth as the preferred direction of propagation, we introduce a representation of the square-root operator as an integral in which a rational function of the transverse Laplacian appears in the integrand. This allows for an explicit asymptotic analysis of the resulting one-way wave equations. We have proven that ray theory for these one-way wave equations leads to one-way eikonal equations and the correct leading order transport equation for the full wave equation. By introducing appropriate boundary conditions at $z=0$, we generate waves at depth whose quotient leads to a reflector map and estimate of the ray-theoretical reflection coefficient on the reflector. Thus, these true amplitude one-way wave equations lead to a "true amplitude wave equation migration (WEM)" method when we use the same imaging condition as is standardly used in WEM. In fact, we have proven that applying the WEM imaging condition to these newly defined wavefields in heterogeneous media leads to the Kirchhoff inversion formula for common-shot data. This extension enhances the original WEM. The objective of that technique was a reflector map, only. The underlying theory did not address amplitude issues. Computer output using numerically generated data confirms the accuracy of this inversion method. However, there are practical limitations. The observed data must be a solution of the wave equation. Therefore, the data over the entire survey area must be collected from a single common-shot experiment. Multi-experiment data, such as common-offset data, cannot be used with this method as presently formulated. Research on extending the method is ongoing at this time.

Introduction

One-way wave equations provide fast tools for modeling and migration. These one-way equations allow us to separate solutions of the wave equation into downgoing and upgoing waves except in the limit of near-horizontal propagation. The original one-way wave equations used for wave equation migration (WEM) [Claerbout, 1971, 1985] were designed to produce accurate traveltimes, but were never intended to produce accurate amplitudes, even at the level of leading order asymptotic WKBJ or ray-theoretic amplitudes. As such, classic WEM provides a reflector map consistent with the background propagation model, but with unreliable amplitude information.

The objective of this presentation is to describe a modification of those one-way wave equations to produce equations that provide accurate leading order WKBJ or ray-theoretic amplitude as well as accurate traveltime. The necessary modification of the basic one-way wave equations is motivated by considering depth-dependent ($v(z)$) medium. In that case, through the use of Fourier transform in time and transverse spatial coordinates (x,y), we reduce the problem of modifying the one-way equations to the study of ordinary differential equations. There, it is relatively simple to see how to modify the equations used by Claerbout in order to obtain equations that provide leading order WKBJ amplitudes, as well. This leading order amplitude is what we mean by "true amplitude" for forward modeling.

For heterogeneous media, $v = v(x,y,z)$, the same one-way wave equations still provide true amplitudes. However, now the transverse wave vector (k_x,k_y) must be interpreted as differentiations in the corresponding dual spatial variables. Further, our modified one-way wave equations involve square-roots and divisions by functions of this transverse wave vector. We provide an interpretation of these operators through some basic ideas from the theory of pseudo-differential operators.

We provide a relatively simple representation of the one-way differential operators. This, in turn allows us to prove that the ray-theoretic solutions of these equations satisfy the separate eikonal equations for downgoing and upgoing waves, but the leading order amplitudes also satisfy the same equation—the *transport* equation—as does the leading order amplitude for the full wave equation. It is in this sense that we designate the solutions of these one-way wave equations as "true amplitude" solutions.

Having these true amplitude one-way equations allows us to develop a "true amplitude'" WEM for heterogeneous media. To date, we only have numerical checks on this method for ($v(z)$) media, where the pseudo-differential operators revert to simple multiplications in the temporal/-transverse-spatial Fourier domain. However, we are able to prove that the reflection amplitude agrees with the amplitudes generated by Kirchhoff inversion (true amplitude Kirchhoff migration) as developed by one of the authors [Bleistein, 1987, Bleistein et al. 2001] and colleagues. This proof is valid in heterogeneous media. Thus, at this time, the proof of validity is ahead of the computer implementation in terms of generality and it anticipates a reliable computer implementation in general heterogeneous media. It confirms that the output of this method is a reflector map with the peak amplitude on the reflector being in known proportion to an angularly dependent reflection coefficient at a specular reflection angle.

This type of inversion requires common-shot data with the receiver array covering the entire domain of the survey. This is a serious obstacle for practical implementation; such data gathers are still relatively rare. To date, we do not have an extension of this true amplitude WEM to other source/receiver configurations.

Dynamically correct one-way wave equations

We begin by considering the wave equation in three spatial dimensions and time:

$$\frac{1}{v^2}\frac{\partial^2 W}{\partial t^2} - \nabla^2 W = 0. \qquad (1)$$

Let us first consider the wave equation in a homogeneous medium and apply Fourier transform in time and in the transverse spatial variables.

$$\mathrm{L}W = \frac{\partial^2 W}{\partial z^2} + k_z^2 W = 0, \qquad (2)$$

where

$$k_z = sign(\omega)\sqrt{\frac{\omega^2}{v^2} - \bar{k}^2} = \frac{\omega}{v}\sqrt{1-\frac{(v\bar{k})^2}{\omega^2}},$$

and

$$\bar{k} = (k_x, k_y), \quad \bar{k}^2 = k_x^2 + k_y^2 .$$

Because the wavespeed is constant, it is possible to transform equation (1) into

$$\mathrm{L}W = \left(\frac{\partial}{\partial z} \mp ik_z\right)\left(\frac{\partial}{\partial z} \pm ik_z\right)W = 0 . \qquad (3)$$

The separate one-way equations implied in (3) have exact solutions that are also solutions of that two-way equation:

$$\left(\frac{\partial}{\partial z} \pm ik_z\right)\left\{A_\pm \exp\{\mp ik_z z\}\right\} = 0, \qquad (4)$$

Here, the A's are constants.

For wavespeed $v(z)$, these solutions are no longer valid. For (2), we would then content ourselves with asymptotic WKBJ solutions. Then, we would want the solutions of the one-way equations to agree, at least asymptotically, with those solutions of the two-way equation (1). For the one-way operators in (4), the exact solutions have the same traveltimes as the WKBJ solutions to (2), but do not have the same amplitude. This leads us to look for ways to modify the two one-way equations in (4), so that the new equations provide a transport equation that yields the same amplitude as in (2).

Let us introduce the scaling

$$\bar{p} = \frac{\bar{k}}{\omega}, \quad p_z = \frac{k_z}{\omega} = \frac{1}{v(z)}\sqrt{1-(v(z)\bar{p})^2} \qquad (5)$$

and rewrite the wave equation in the form

$$\frac{\partial^2 W}{\partial z^2} + \omega^2 p_z^2 W = 0 . \qquad (6)$$

Now, we assume a solution to (6) of the form

$$W = A(z,\bar{p})e^{i\omega\tau(z,\bar{p})} \qquad (7)$$

and write down the eikonal and transport equations,

$$\left(\frac{d\tau}{dz}\right)^2 = p_z^2 \;\Rightarrow\; \frac{d\tau}{dz} = \pm p_z, \qquad (8)$$

and
$$2\frac{d\tau}{dz}\frac{dA}{dz} + \frac{d^2\tau}{dz^2}A = 0$$

or
$$\pm\left[2p_z\frac{dA}{dz} - \frac{1}{v^3(z)p_z}\frac{dv(z)}{dz}A\right] = 0$$

or
$$\frac{dA}{dz} - \frac{1}{2v^3(z)p_z^2}\frac{dv(z)}{dz}A = 0 . \qquad (9)$$

Substituting the same form of solution in (6) into the one-way equations (4) yields the two branches of the eikonal equation, but has as transport equation only $dA/dz = 0$. The last form of the transport equation in (9) suggests terms to add to the two equations in (4) in order to obtain one-way wave equations with the right amplitude—the right dynamics—for this $v(z)$ case. We first transform (4) to the variables introduced in (5) and then modify those equations to

$$\left[\frac{\partial}{\partial z} \mp i\omega p_z - \frac{1}{2v^3(z)p_z^2}\frac{dv(z)}{dz}\right]W = 0 . \qquad (10)$$

These are our modified one-way wave equations. In the original variables, they become

$$\left[\frac{\partial}{\partial z} \mp ik_z - \frac{\omega^2}{2v^3(z)k_z^2}\frac{dv(z)}{dz}\right]W = 0. \qquad (11)$$

For $v(z)$ media, these one-way wave equations produce the same eikonal equation and transport equation as does the two-way wave equation, (1).

For heterogeneous media in which $v = v(x, y, z)$, traditional WEM continues to use the one-way wave operators in (3). The transverse wave vectors in those equations are interpreted as derivatives in the transverse directions and various rational approximations are made to avoid the square-root of differential operators implicit in the representation. In fact, interpretation and manipulation of such operators is a major component of the theory of pseudo-differential operators. Through proper interpretation, these operators can be analyzed rigorously. In G. Q. Zhang [1993], the second author has done just this. To explain, we think of the transverse wave vector and frequency in (11) as being symbolic place-holders for differentiations:

$$i\omega \leftrightarrow -\partial/\partial t \quad \text{and} \quad i(k_x, k_y) \leftrightarrow \nabla(\partial/\partial x, \partial/\partial y).$$

We then think of ik_z as a *symbol* for a differential operator. More precisely, we rewrite (11) as

$$\mathrm{L}_{\pm} W = \left[\frac{\partial}{\partial z} \pm \Lambda\right] W - \Gamma W = 0 \tag{12}$$

with Λ and Γ being pseudo-differential operators with symbols, Λ and Γ given by

$$\Lambda = ik_z = i\frac{\omega}{v}\sqrt{1 - \frac{(v\bar{k})^2}{\omega^2}},$$

$$\Gamma = -\frac{1}{2k_z}\frac{\partial k_z}{\partial z} = \frac{v_z}{2v}\left[1 + \frac{(v\bar{k})^2}{\omega^2 - (v\bar{k})^2}\right]. \tag{13}$$

Here, it is easy to check that the expression for Γ is exactly the same as the last factor in (11).

In completely heterogeneous media—$v = v(x, y, z)$—G. Q. Zhang has shown that the eikonal equations for (11) provide the two branches of the eikonal equation for (1). Furthermore, transport equations for (11) are both the same as the transport equation for (1). Thus, he has proven that the one-way wave equations (11) provide the dynamics as well as the kinematics of the full wave equation.

His proof was greatly facilitated by an important identity that he derived, namely, that

$$\Lambda = ik_z = i\frac{\omega}{v}\left[1 - \frac{1}{\pi}\int_0^1 \sqrt{1 - s^2}\,\frac{(v\bar{k})^2}{\omega^2 - s^2(v\bar{k})^2}\,ds\right]. \tag{14}$$

This identity interprets the square root operator as a rational quotient of operators with the square root appearing only in the integration variable. The divisions here and in the second part of (13) can be interpreted as inverse wave operators, which we might think of as convolutions with appropriate Green's functions. Alternatively, we could introduce an auxiliary function through the equation

$$\left[\frac{\partial^2}{\partial t^2} - s^2(v\nabla_{Tx})^2\right] q(s; x, z, t) = (v\nabla_{Tx})^2 W(x, z, t) \tag{15}$$

and then rewrite (12) as

$$\frac{\partial W}{\partial z} \pm \frac{1}{v}\frac{\partial W}{\partial t} \mp \frac{1}{\pi v}\frac{\partial}{\partial t}\int_0^1 \sqrt{1 - s^2}\,q(s; x, z, t)\,ds + \frac{v_z}{2v}\left[W + q(1; x, z, t)\right] = 0. \tag{16}$$

Through this device of introducing the identity in (14) and the auxiliary function in (15), we are able to interpret the pseudo-differential operators in (12) in terms of ordinary operators and ordinary solutions of differential equations. The proof that the correct eikonal and transport equations arise from (12) then becomes a fairly straightforward if somewhat tedious calculation.

True amplitude wave equation migration

Here we describe our proposed true amplitude WEM motivated by the dynamically correct one-way wave equations of the previous section. We begin by introducing Claerbout's [1971,1985] classic WEM and explain how we modify the governing equations and boundary data to obtain our proposed true amplitude WEM.

The standard method uses the one-way propagators of (4), even for heterogeneous media. More specifically, suppose that the reflected wave field from a single source experiment is observed at $z = 0$ for all time. Then the source and observed wavefields are assumed to be solutions of the equations

$$\begin{cases} \left[\dfrac{\partial}{\partial z} + \Lambda\right] D = 0, \\ D(x, y, z = 0; \omega) = \delta(\vec{x} - \vec{x}_S), \end{cases} \tag{17}$$
$$\vec{x} = (x, y, z), \quad \vec{x}_s = (x_s, y_s, 0),$$

and

$$\begin{cases} \left[\dfrac{\partial}{\partial z} - \Lambda\right] U = 0, \\ U(x, y, z = 0; \omega) = Q(x, y; \omega), \end{cases} \tag{18}$$

where D is the downgoing (source) wavefield and U is the upgoing wavefield with the observed data Q. The image is then produced as an *impedance* or *reflectivity* function at every image point defined by

$$R(\vec{x}) = \int \frac{U(\vec{x}; \omega)}{D(\vec{x}; \omega)}\,d\omega. \tag{19}$$

While this result produces a reflector map, it does not provide accurate amplitude information. To achieve that, we use the solutions of our modified true amplitude one-way wave equations, (12). That is, we introduce p_D and p_U as solutions of the following problems.

$$\begin{cases} \left[\dfrac{\partial}{\partial z} + \Lambda - \Gamma\right] p_D = 0, \\ p_D(x, y, z = 0; \omega) = -\dfrac{1}{2}\Lambda^{-1}\delta(\vec{x} - \vec{x}_s), \end{cases} \tag{20}$$

$$\left\{\begin{array}{l} \left[\dfrac{\partial}{\partial z} \;\square\square\;\square\square\square\right] p_U = 0, \\ p_U(x,y,z=0;\omega) = Q(x,y;\omega). \end{array}\right. \qquad (21)$$

Here, we have modified the equations and we have also modified the boundary data for p_D. The reason is that this is the proper data to model a point source for the original wave equation. Note that this modification involves both a scaling and phase shift because of the i in the definition of $\square$ in (13). Next, we modify the imaging condition (18) to use our new wavefields. That is, we set

$$R(\vec{x}) = \int \frac{p_U(\vec{x};\omega)}{p_D(\vec{x};\omega)} d\omega \,. \qquad (22)$$

See Zhang et al. [2001,2002].

Comparison of true amplitude WEM and Kirchhoff inversion.

Relying on G. Q. Zhang's [1993] proof of the equivalence of the solutions of the one-way wave equations with the solutions of the full wave equation, we derive the asymptotic form of (20) and (21) in terms of the traveltimes and amplitudes of the full wave equation. That result is

$$R(\vec{x}) = 2\iint i\omega \frac{\cos\theta_r}{v_r} \frac{A(\vec{x}_r;\vec{x})}{A(\vec{x};\vec{x}_s)} e^{i\omega[\tau(\vec{x}_r;\vec{x})+\tau(\vec{x};\vec{x}_s)]} dx_r dy_r \,. \qquad (23)$$

Here, v_r is the wave speed at the receiver point and θ_r is the emergence angle of the ray from the image point to the receiver point. Furthermore, the amplitudes and phases are solutions of the eikonal and transport equations for the full wave equation. Their equivalence with the solutions of the one-way wave equations is a consequence of G. Q. Zhang's proof. This is the formula for common shot Kirchhoff inversion in Bleistein [1987] and Bleistein et al. [2001] as expressed by Hanitzsch [1997].

In summary, then, we have proven that the one-way wave equations introduced in (12) produce WKBJ solutions that agree with the solutions of the full wave equation (1) both kinematically and dynamically. Further, we have shown that if we use these equations and appropriate boundary data derived from the common-shot seismic experiment, the WEM imaging condition leads to the Kirchhoff common-shot inversion formula. An important difference, however, is that the solutions generated from the one-way wave equations include all multi-pathing arrivals at the image point, whereas this is problematic for Kirchhoff methods that only use the simplest WKBJ Green's functions.

Numerical test

To show how true amplitude common-shot migration works, we apply it to a 2-D horizontal reflector model in a medium with velocity $v = 2000 + 0.3z$. Recall from the theory that in this case, the modeling and migration can be carried out in the transverse spatial and temporal Fourier domains, with (k_x, k_y) being the simple transverse part of the wave vector.

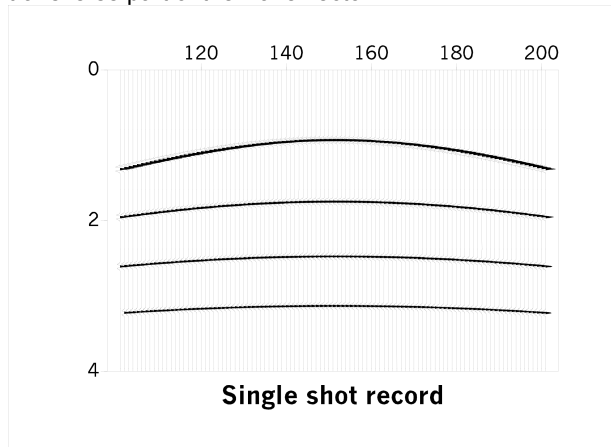

Figure 1: Single shot data record

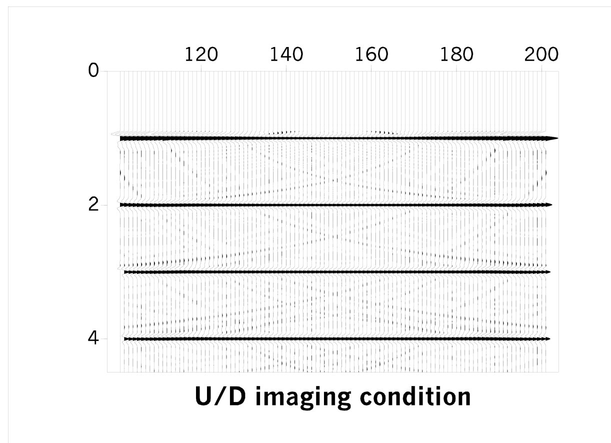

Figure 2: Finite difference migration using (18) for the imaging condition

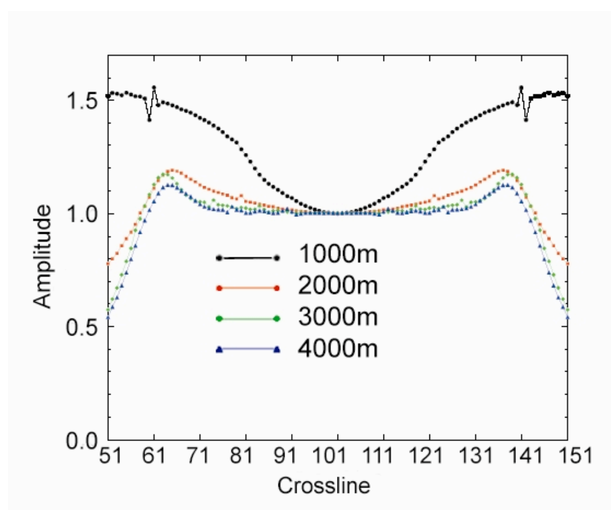

Figure 3: Peak amplitudes along the four reflectors. The wide angle error decreases with depth of the reflector.

The input data (Figure 1) is a single shot record over four horizontal reflectors from density contrast. Figure 2 shows the migrated shot record using the conventional common-shot migration algorithm (19). The peak amplitudes along the four migrated reflectors are shown in Figure 3, normalized to the geometrical optics reflection coefficient along the reflector. This method has a phase error as noted above. The consequent phase error has been corrected during the migration. However, the migrated amplitudes are poor, especially on the reflector at depth $z = 1000m$ along which the reflection angles vary over a wide range. (This method has incorrect angular dependence when compared to true amplitude reflectivity or the geometrical optics reflection coefficient at each point.) The wide angle peak amplitudes decrease monotonically with increasing depth. The greatest error occurs at wide angle, with the result along the shallowest reflector being the worst. However, the error is zero at zero offset; in this limit, $(k_x, k_y) = (0,0)$ and $\cos\theta_r = 1$.

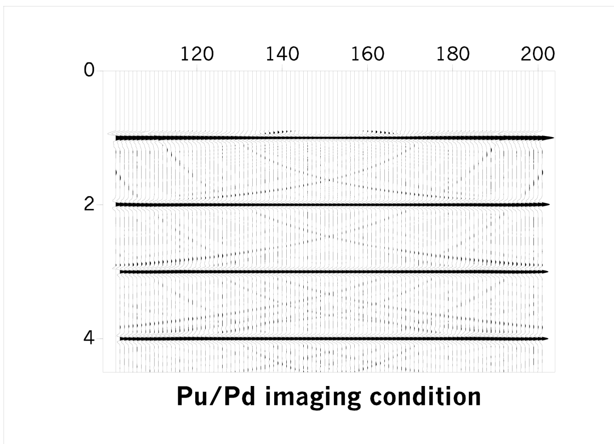

Figure 4: Finite difference migration using (20) for imaging.

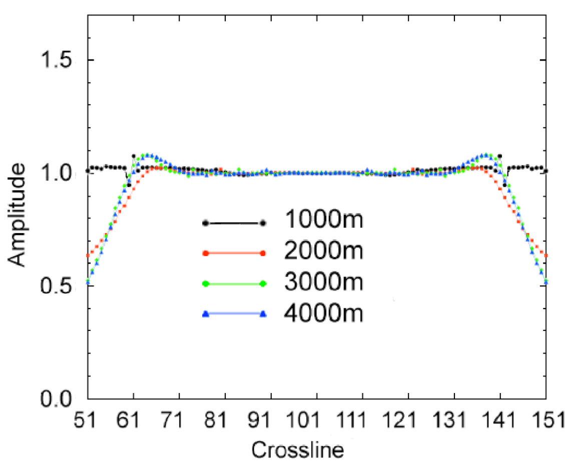

Figure 5: Peak amplitudes along the four reflectors. The wide angle error decreases with depth of the reflector.

Figure 4 shows results of true amplitude common-shot migration using (23). The peak amplitudes along the reflectors are shown in Figure 5. From this plot, we clearly see that the true amplitude algorithm recovers the reflectivity accurately, aside from the edge effects and small jitters caused by interference with wraparound artifacts.

Conclusions

Common-shot migrations offer good potential of imaging complex structures, but the conventional formulations of such migrations produce incorrect migrated amplitudes. Here, we have described true-amplitude one-way wave equations that allow us to extend the standard method both for forward modeling and for wave equation migration. These modified one-way wave operators are developed with the aid of pseudo-differential operator theory. We have proven that these new one-way wave equations provide solutions that agree dynamically, as well as kinematically, with the solutions of the full wave equation. Further, we have proposed a new approach to WEM, transforming it into a true amplitude process, meaning that it produces an inversion output that agrees asymptotically with Kirchhoff inversion: it produces a reflector map with peak amplitudes on the reflector in known proportion to the geometrical optics reflection coefficient. We have proven this claim, as well. With the aid of a simple numerical example, we demonstrated that the migration method we proposed does calibrate common-shot migrations by correcting both their amplitude and phase behavior. We did this for an example in which the wave speed is depth-dependent—$v = v(z)$. The new method actually builds a bridge between true amplitude common-shot Kirchhoff migration and the migrations based on one-way wavefield extrapolation.

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