Series expansion of wave propagation in heterogeneous media using least-squares

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ABSTRACT

In seismic modeling and imaging, one-way operators are used to estimate wavefields in the subsurface by extrapolating recorded wavefields. The most general one-way operator for isotropic media takes the form of a pseudo-differential operator that is a function of vertical slowness q, and the vertical distance Δz between the recording location and the source location. It is assumed that q varies in the lateral coordinates x and y, but is invariant over Δz.

The general extrapolation operator is still only an approximation, and we use least squares to improve accuracy. Using a seismic modeling algorithm, we implement the least-squares operator and extrapolate a point source through a strongly heterogeneous medium. We compare the resulting wavefield to wavefields to the expected wavefield obtained analytically. We find that the least-squares approach provides a superior result to the general operator and its adjoint but for much higher computational cost. We derive a series approximation to the least-squares operator suggesting that suggests for minimal loss of accuracy, large increases in computational efficiency can be realized.

INTRODUCTION

Wavefield extrapolation forms the basis of many imaging and modeling software used in exploration seismology. The best extrapolators balance accuracy and efficiency and understanding this balance is critical to imaging success. The basic data of seismology are the recorded seismic wavefields:

In a seismic array, geophones are located relative to three orthogonal axes x_1, x_2 and z. Two of them, x_1 and x_2 represented in short by x, define a plane normal to the approximate direction of the Earth's gravity near the center of the geophone array. Orthogonal to x, axis z represents depth into the Earth.

We will represent the seismic waves recorded in the geophone array by $\psi(x,z,t)$, where t is an axis representing the passage of time since excitation of the source. Fourier transform of ψ is

$$\psi(x,z,\omega) = \frac{1}{2\pi}\int \psi(x,z,t)\, e^{i\omega t}\, dt, \tag{1}$$

where ω is temporal frequency, and the limits of integration are defined by the compact support of t. Subsequent operations on ψ are general in ω so the ω notation is omitted hereafter.

If we have a notion of the variability and anisotropy of the subsurface, wavefield ψ in the subsurface can be predicted using ψ recorded at the surface.

EXTRAPOLATION OPERATOR $\mathbf{P}$

Imaging and modeling algorithms based on extrapolation operators are robust in the presence of large errors in velocity (they don't generate internal reflections and mode conversions) and they are not restricted to traveltime assumptions (they allow multipathing).

Wavefield ψ can be extrapolated from z at the surface to $z+\Delta z$ in the subsurface using (Margrave and Ferguson, 1999)

$$[\mathbf{P}(\Delta z)\,\psi(y,z)](x,z+\Delta z) = \frac{1}{(2\pi)^2}\int e^{-i\langle x-y,\xi\rangle} c(x,\xi,\Delta z)\,\psi(y,z)\,d\xi dy, \tag{2}$$

where

$$c(x,\xi,\Delta z) = e^{i\Delta z\omega q(x,\xi)}. \tag{3}$$

Operator $\mathbf{P}$ transforms input $\psi(y)$ from input coordinates (y,z) to output coordinates $(x,z+\Delta z)$, and q represents the heterogeneity of the anisotropic slowness between z and $z+\Delta z$.

$\mathbf{N}$ THE ADJOINT OF $\mathbf{P}$

A second useful operator is $\mathbf{N}$ the adjoint of $\mathbf{P}$ given by

$$[\mathbf{N}(-\Delta z)\,\psi(x,z)](y,z-\Delta z) = \frac{1}{(2\pi)^2}\int e^{i\langle x-y,\xi\rangle} c(x,\xi,-\Delta z)\,\psi(x,z)\,d\xi dy \tag{4}$$

The equations for $\mathbf{N}$ and $\mathbf{P}$ differ only in the signs on Δz and the exponent of the Fourier kernal, and on the spatial dependence of c - $\mathbf{P}$ varies with output coordinate x, and $\mathbf{N}$ varies with input coordinates y. The impulse responses for $\mathbf{N}$ and $\mathbf{P}$, however, are significantly different as demonstrated in Figure 1.

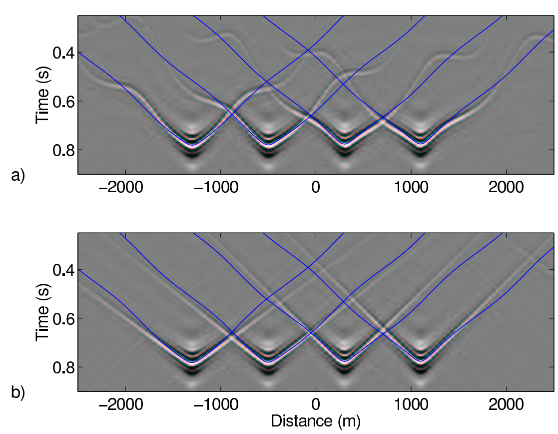

Figure 1: Impulse responses with analytic impulse response plotted. (a) $\mathbf{P}$, large error relative to analytic impulse response. (b) $\mathbf{N}$, large error relative to analytic impulse response. The velocity used is a sinusoid varying laterally between 1000 and 3000 m/s. The extrapolation interval is 300m.

A LEAST-SQUARES EXTRAPOLATOR

As demonstrated in Figure 1, $\mathbf{P}$ and $\mathbf{N}$ are inaccurate where there is strong lateral velocity variation. As a remedy, a least-squares operator can be derived based on the following equality:

$$\mathbf{P}\left(\frac{\Delta z}{2}\right)\psi(z) = \mathbf{P}\left(-\frac{\Delta z}{2}\right)\psi(z+\Delta z). \tag{5}$$

Following the standard least-squares proceedure, multiply equation (5) by the adjoint of $\mathbf{P}\left(-\frac{\Delta z}{2}\right)$ and invert:

$$\psi(z+\Delta z) = \left[\mathbf{N}\left(\frac{\Delta z}{2}\right)\mathbf{P}\left(-\frac{\Delta z}{2}\right)\right]^{-1}\mathbf{N}\left(\frac{\Delta z}{2}\right)\mathbf{P}\left(\frac{\Delta z}{2}\right)\psi(z). \tag{6}$$

The impulse response for the least-squares operator in Figure 2b is an improvement over both $\mathbf{P}$ and $\mathbf{N}$. (Figure 2a is the least-squares operator with the inversion term set to identity for comparison).

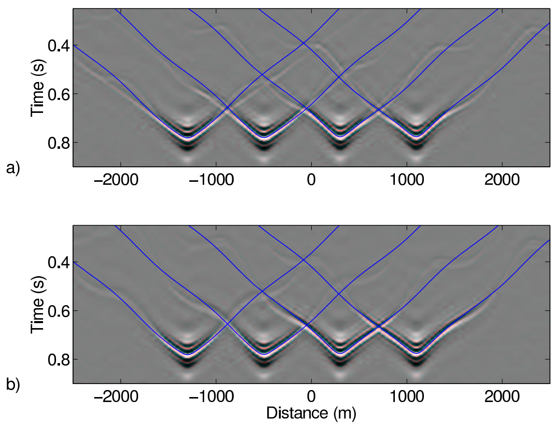

Figure 2: Impulse responses of the least-squares operator with analytic impulse response plotted. (a) For a fast operator, assume $\left[\mathbf{N}^{+}\mathbf{P}^{-}\right]^{-1} = \mathbf{I}$. The operator still damps the erroneous amplitudes compared to Figures 1a and 1b. (b) Full computation of $\left[\mathbf{N}^{+}\mathbf{P}^{-}\right]^{-1}$, provides improved damping when compared to (a) but is much more expensive.

SERIES FOR NP

The matrix multiplications and inversion required by the least-squares operator are computationally prohibitive - especially in 3D. Fast alternatives to the cascade of operators $\mathbf{N}$ and $\mathbf{P}$ are desirable. To this end we may seek a series expansion for $\mathbf{NP}$. To begin, cast $\mathbf{NP}$ entirely in the Fourier domain

$$\left[\mathbf{N}\left(\frac{\Delta z}{2}\right)\mathbf{P}\left(\frac{\Delta z}{2}\right)\varphi(\eta,z)\right](\xi,z+\Delta z) = \frac{1}{(2\pi)^2}\int \varphi(\eta,z)\,c\left(x,\eta,\frac{\Delta z}{2}\right)c\left(x,\xi,\frac{\Delta z}{2}\right)e^{i\langle x,\xi-\eta\rangle}dx\,d\eta, \tag{7}$$

where

$$\varphi(\eta,z) = \int \psi(y,z)\,e^{i\langle\eta,y\rangle}dy. \tag{8}$$

Write $c\left(x,\eta,\frac{\Delta z}{2}\right)$ as a Taylor series in $c\left(x,\xi,\frac{\Delta z}{2}\right)$ and $\mathbf{NP}$ becomes

$$\left[\mathbf{N}\left(\frac{\Delta z}{2}\right)\mathbf{P}\left(\frac{\Delta z}{2}\right)\varphi(\eta,z)\right](\xi,z+\Delta z) = \sum_{m=0}^{\infty}\int\frac{1}{m!}\partial_{\xi}^{m}c\left(x,\xi,\frac{\Delta z}{2}\right)c\left(x,\xi,\frac{\Delta z}{2}\right)i^{m}\partial_{x}^{m}\left[\psi(x,z)\,e^{i\langle\xi,x\rangle}\right]dx \tag{9}$$

Integrate equation (9) by parts to get

$$\left[\mathbf{N}\left(\frac{\Delta z}{2}\right)\mathbf{P}\left(\frac{\Delta z}{2}\right)\psi(x,z)\right](\xi,z+\Delta z) = \int\psi(x,z)\,\beta(x,\xi,\Delta z)\,e^{i\langle\xi,x\rangle}dx, \tag{10}$$

where

$$\beta(x,\xi,\Delta z) = \sum_{m=0}^{\infty}\frac{i^m}{m!}\,\partial_x^m\left[c\left(x,\xi,\frac{\Delta z}{2}\right)\partial_{\xi}^{m}c\left(x,\xi,\frac{\Delta z}{2}\right)\right]. \tag{11}$$

Cast entirely in the space domain, $\mathbf{NP}$ becomes

$$\left[\mathbf{N}\left(\frac{\Delta z}{2}\right)\mathbf{P}\left(\frac{\Delta z}{2}\right)\psi(x,z)\right](y,z+\Delta z) = \frac{1}{(2\pi)^2}\int\psi(x,z)\,\beta(x,\xi,\Delta z)\,e^{i\langle\xi,x-y\rangle}d\xi dx. \tag{12}$$

A series of a few well chosen terms costs significantly less computationally than the general operator. For example, in 3D, a series with a single term requires two integrations over the space coordinates where the general form requires four.

FACTOR SYMBOL C FOR FAST EXTRAPOLATION

Even using a single term, implementation of $\mathbf{NP}$ is costly due to the integrations. A way to speed up $\mathbf{NP}$ is to specify symbol c such that integration is replaced by fast Fourier transform. Also, there is the problem of the matrix inverse, namely $\left[\mathbf{N}^{+}\mathbf{P}^{-}\right]^{-1}$.

Assume that c is separable such that

$$c(x,\xi) = a(x)\,b(\xi), \tag{13}$$

with a and b given by

$$a(x,\Delta z/2,\omega) = e^{i\frac{\Delta z}{2}\omega\hat{q}(x,\omega)}, \tag{14}$$

and

$$b(\xi,\Delta z/2,\omega) = e^{i\frac{\Delta z}{2}\omega\bar{q}(\xi,\omega)}. \tag{15}$$

Then β corresponding to $\mathbf{N}(\Delta z/2)\,\mathbf{P}(-\Delta z/2)$ is simply

$$\beta_{\mathbf{N}^{+}\mathbf{P}^{-}} = 1. \tag{16}$$

This significant result suggests that, for $c(x,\xi) = a(x)\,b(\xi)$, neither $\mathbf{N}^{+}\mathbf{P}^{-}$ nor it's inverse need be computed for the least-squares operator.

Using $c(x,\xi) = a(x)\,b(\xi)$, β corresponding to $\mathbf{NP}$ becomes

$$\beta_{\mathbf{NP}}(x,\xi) = a^2(x)\,b^2(\xi) + b(\xi)\sum_{m=1}^{\infty}\frac{i^m}{m!}\partial_x^m a^2(x)\,\partial_{\xi}^{m}b(\xi), \tag{17}$$

and extrapolation proceeds as follows:

1. For each $0 \le m \le \infty$, terms corresponding to a are multiplied by input wavefield ψ.
2. Because only $e^{i\langle\xi,x-y\rangle}$ has space/wavenumber dependence, FFT $x \to \xi$
3. Multiply by the corresponding b terms.
4. IFFT $\xi \to y$

With factorization of c, the cost of extrapolation is proportional to the number of terms in β times the cost of the fast Fourier transform. This provides a significant saving especially in 3D. Finite differences should probably used to compute $\partial_x^m a(x)$.

VELOCITY SMOOTHNESS, HIGH FREQUENCY, AND STABILITY

Only $\partial_\xi^m b(\xi)$ in the series part of $\beta_{\mathbf{NP}}$ is analytic. Symbol a represents the heterogeneity in the medium. Symbol b represents a background medium that may be anisotropic and is usually analytic. Analysis of $\partial_\xi^m b(\xi)$ suggests that only even terms are significant. Odd terms are scaled by ξ and IFT ($\xi \rightarrow x$) annihilates them. In general:

$$\partial_\xi^m b(\xi, \Delta z/2, \omega) = i\frac{\Delta z}{2}\omega b(\xi, \Delta z/2, \omega)\,\partial_\xi^{2m}\bar{q}(\xi, \omega), \tag{18}$$

and

$$\begin{aligned}&\beta_{\mathbf{NP}}(x, \xi, \Delta z/2, \omega) = \\ &a^2(x, \Delta z/2, \omega)\, b^2(\xi, \Delta z/2, \omega) \\ &+ i\frac{\Delta z}{2}\omega b(\xi, \Delta z/2, \omega)\sum_{m=1}^{\infty}\frac{i^{2m}}{(2m)!}\partial_x^{2m} a^2(x, \Delta z/2, \omega)\,\partial_\xi^{2m}\bar{q}(\xi, \omega)\end{aligned} \tag{19}$$

For conventional seismic frequency bands (0 - 1 kHz) and velocities (0.1 - 10 km/s), this series for $\beta_{\mathbf{NP}}$ diverges unless a varies smoothly (velocity variation is smooth). Implicit is a high frequency approximation.

EXAMPLES

In seismic imaging, useful prescriptions for $\hat{q}$ and $\bar{q}$ are associated with phase-screen operators. Consistent with the factorization of c we have

$$\hat{q}(x, \omega) = \frac{c_0}{\omega^2}\left[\left(\frac{\omega}{c(x)}\right)^2 - \left(\frac{\omega}{c_0}\right)^2\right], \tag{20}$$

and

$$\bar{q}(\xi, \omega) = \frac{1}{\omega}\sqrt{\left(\frac{\omega}{c_0}\right)^2 - \langle\xi, \xi\rangle}. \tag{21}$$

Vertical slownesses $\hat{q}$ and $\bar{q}$ correspond to the simplest of a class of operators collectively called the *generalized-screen operators* (Wu and Huang, 1992; Le Rousseau and de Hoop, 2001). Velocity c_0 is a background velocity representative of the medium. For the following examples, we set $m = 0$. Propagation of source impulses is simulated for two strongly heterogeneous velocity models (isotropic, Figures 3 and 5). The SEG salt model (Figure 3) represents a salt body embedded in a smoothly varying sedimentary background. A source is forward modeled and a snapshot is taken in depth and distance (Figure 4). For comparison, the analytic impulse response from traveltime calculations is plotted. Good agreement of first arrivals can be seen with major departures associated with headwaves (the sides of the impulse response) not extrapolated by extrapolation operators, and a caustic (beneath the salt) that the travel time calculation cannot model. Similar results are apparent for the Marmousi model (Figures 5 and 6). Comparison to finite difference modeling is required to establish the validity of multipathed arrivals in both figures.

CONCLUSIONS

A new least-squares approach is used to derive a series representation for wavefield extrapolation. Terms in this series consist of spatial and wavenumber derivatives of the extrapolation symbol. An approximation to the extrapolation symbol us used to significantly reduce the computational load implied by the least-squares extrapolator. It is based on factoring the extrapolation symbol into a product of two terms. One term representing the lateral heterogeneity of the isotropic approximation to the medium. The other representing a background medium that

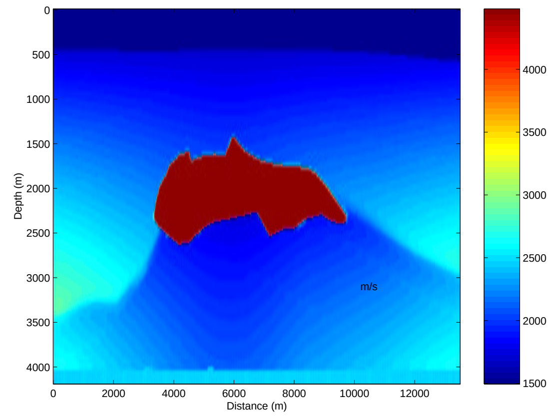

Figure 3: SEG salt model. Velocities correspond to P-waves traveling in an isotropic medium

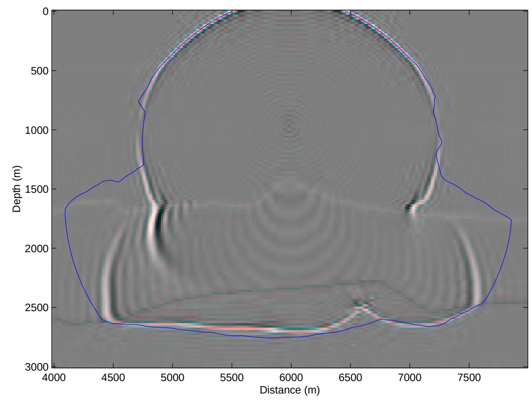

Figure 4: Snapshot of a wavefield propagating through the model of Figure 3. A standard, first-order phase-screen operator (Wu and Huang, 1992; Le Rousseau and de Hoop, 2001) was used to compute this image. The location of the source is 6000m distance and 1000m depth, and the elapsed time is 0.59ms. For comparison, the corresponding traveltime contour is plotted. First arrivals reliably track the analytic response with departures due to headwave generation (one-way extrapolation operators don't model head waves) and multipathing (traveltime methods have limited ability to model multipathing).

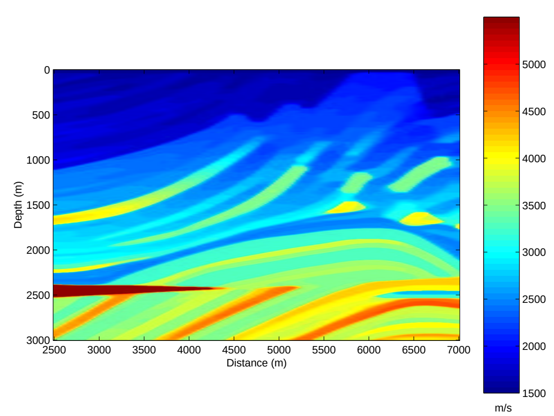

Figure 5: The Marmousi model. Velocities correspond to P-waves in an isotropic medium.

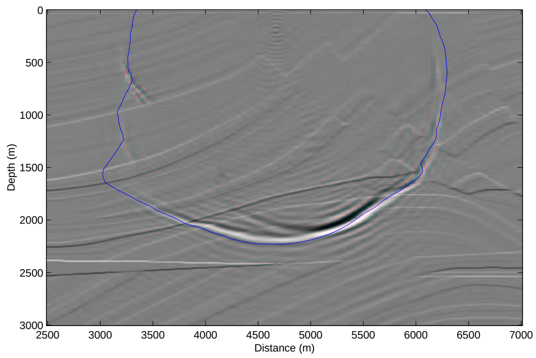

Figure 6: Snapshot of a wavefield propagating through the model of Figure 5. A standard, first-order phase-screen operator (Wu and Huang, 1992; Le Rousseau and de Hoop, 2001) was used to compute this image. The location of the source is 6000m distance and 1000m depth, and the elapsed time is 0.59ms. For comparison, the corresponding traveltime contour is plotted. First arrivals reliably track the analytic response with departures due to headwave generation (one-way extrapolation operators don't model head waves) and multipathing (traveltime methods have limited ability to model multipathing).

can be anisotropic. The result eliminates the requirement of computing two matrix multiplications and a matrix inverse in the least-squares approach.

For velocity variation that is not smooth, the extrapolation symbol was found to be stable in the high frequency limit. Two forward modeling examples were presented to demonstrate the utility of this operator. The 0^{th} order version of the extrapolation symbol was used and found to be identical to the first order phase-screen operator. This operator returned very good results for first arrivals compared to forward modeling using traveltime computation, and provided multipathed arrivals as well.

For smooth velocity variation, the least-squares operator is an enhancement to the class of phase-screen operators, allowing them to be optimal in a least-squares sense for an extra cost proportional to the number of terms chosen for the extrapolation symbol. It is hoped that new approaches to truncation of this series will result in fast extrapolation operators that are robust in the presence of strong velocity variation.

REFERENCES

Le Rousseau, J. H., and de Hoop, M. V., 2001, Modeling and imaging with the scalar generalized-screen algorithms in isotropic media: Geophysics, **66**, 1551–1568.

Margrave, G. F., and Ferguson, R. J., 1999, Wavefield extrapolation by nonstationary phase shift: Geophysics, **64**, 1067 – 1078.

Wu, R. S., and Huang, L. Y., 1992, Scattered field calculation in heterogeneous media using phase-screen propagation: Scattered field calculation in heterogeneous media using phase-screen propagation:, Soc. of Expl. Geophys., 62nd Ann. Internat. Mtg, 1289–1292.