# Impedance-Type Inversion of the P-P Reflection Coefficient

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## Abstract

The exact expression for the P-P reflection coefficient (Zoeppritz equations) has a rather complicated dependence on the medium parameters (P- and S-wave velocities and density) at both sides of the interface. For the inversion purpose, it's required approximations formulas. In this work we discuss some approximations based on Taylor series and on the impedance concept. We also present the inversion process induced by these approximations.

## Introduction

With some assumptions in the model parameters, like restrictions on the elastic parameters contrast, or at the incidence angle, approximations to the Zoeppritz equation, based on Taylor series, that have simpler expression and provide better access to the medium parameters can be derived. From these expressions we can establish a simple inversion process based on linear least square method. However this technique is not so efficient, because it doesn't provide an explict expression for recuperate the elastic parameters, or their contrasts. Using this, we just obtain linear combinations of the contrast of these parameters.

Recently, some authors like Connoly (1999) and Santos et al. (2002), showed that approximations for the P-P reflection coefficient and inversion processes using impedance concept provide better results. The ideia of this kind of approximation is to look for a representation similar to the equation for the normal-inicidence elastic reflection coefficient.

## Approximations by Taylor Series

The P-P elastic reflection coefficient $(R_{PP})$ can be approximated by expanding some terms of the Zoeppritz equation in a Taylor series, with some assumptions in the model parameters. In the following, we review four of these approximations.

The first one is called *weak-contrast approximation*, which assumes that the contrast in the elastic parameters are small. The linear approximation of Aki & Richards (1980) is given by:

$$R_{PP} \approx \left[\frac{1}{2} - 2\left(\frac{\beta}{\alpha}\right)^2 \sin^2\theta\right]\frac{\Delta\rho}{\rho} + \frac{1}{2}\sec^2\theta\frac{\Delta\alpha}{\alpha} - 4\left(\frac{\beta}{\alpha}\right)^2 \sin^2\theta\frac{\Delta\beta}{\beta}. \quad (1)$$

Here, $\Delta\alpha$ and $\alpha$ denote, respectively, the average and the difference of the P-wave velocity values, $\alpha_1$ and $\alpha_2$. The same notation is used for S-wave velocity, $\beta$, and density, $\rho$. Moreover, $\theta$ is the incidence angle.

Another way to approximate $R_{PP}$ is to assume that the distance between the source and the receiver is small. Consequently, the ray parameter $p = \sin\theta/\alpha_1$ will be also small. So the *small-offset approximation* is obtained by expanding the square-root terms $(1-v^2p^2)^{\frac{1}{2}}$ in the Zoeppritz equation in a Taylor series. The result is (see, e.g., Täjland (1993)):

$$\begin{aligned} R_{PP} \approx\ & \frac{\rho_2\alpha_2 - \rho_1\alpha_1}{\rho_2\alpha_2 + \rho_1\alpha_1} + \left\{ \frac{\rho_1\alpha_1\rho_2\alpha_2(\alpha_2^2 - \alpha_1^2)}{(\rho_2\alpha_2 + \rho_1\alpha_1)^2} \right. \\ & + \frac{2\alpha_1}{(\rho_2\alpha_2 + \rho_1\alpha_1)^2(\rho_2\beta_2 + \rho_1\beta_1)}\Big[4\rho_1(\rho_2\beta_2^2 + \rho_1\beta_1^2)^2 \\ & - \rho_1\alpha_2^2\beta_1\beta_2(\rho_2 - \rho_1)^2 \\ & \left. - 4\rho_1\rho_2\alpha_2(\rho_2\beta_2^2 + \rho_1\beta_1^2)(\beta_2 + \beta_1)\Big]\right\}p^2. \quad (2) \end{aligned}$$

The third approximation to be presented is the *weak-contrast and small-offset approximation* that requires the two previous assumptions. This is done by expanding all terms of the weak-contrast approximation (1) in a Taylor series in $p$,

$$\begin{aligned} R_{PP} \approx\ & \frac{1}{2}\left(\frac{\Delta\alpha}{\alpha} + \frac{\Delta\rho}{\rho}\right) + \frac{1}{2}\left[\frac{\Delta\alpha}{\alpha} \right. \\ & \left. - 4\frac{\beta^2}{\alpha^2}\left(\frac{\Delta\rho}{\rho} + 2\frac{\Delta\beta}{\beta}\right)\right]\alpha^2p^2 + \frac{1}{2}\frac{\Delta\alpha}{\alpha}\alpha^4p^4. \quad (3) \end{aligned}$$

Finally, the last expression to $R_{PP}$ obtained by Taylor series is the *pseudo-$p^2$ approximation*. According to Wang (1999), a peudo-$p^2$ approximation can be derived by expanding the p-terms in the Zoeppritz equations, in a different way than on the small offset approximation. After some algebraic manipulation the following quadratic expression with respect to elastic contrasts is obtained

$$\begin{aligned} R_{PP} \approx\ & \left[\frac{1}{2} - 2\left(\frac{\beta}{\alpha}\right)^2 \sin^2\theta\right]\frac{\Delta\rho}{\rho} \\ & + \frac{1}{2}\sec^2\theta\frac{\Delta\alpha}{\alpha} - 4\left(\frac{\beta}{\alpha}\right)^2\sin^2\theta\frac{\Delta\beta}{\beta} \\ & + \left(\frac{\beta}{\alpha}\right)^3\cos\theta\sin^2\theta\left(\frac{\Delta\rho}{\rho} + 2\frac{\Delta\beta}{\beta}\right)^2. \quad (4) \end{aligned}$$

In order to analyse the accuracy of the approximations described above we consider two two-layer models which have small and large contrast. Table 1 sumarizes the parameters.

| Medium | $\alpha$(km/s) | $\beta$(km/s) | $\rho$(km/s) |
|---|---|---|---|
| 1 | 3.42 | 1.78 | 2.53 |
| 2 | 3.39 | 1.79 | 2.50 |
| Contrast | 0.01 | 0.01 | 0.01 |
| 1 | 2.77 | 1.52 | 2.30 |
| 2 | 4.55 | 2.61 | 2.44 |
| Contrast | 0.34 | 0.48 | 0.02 |

Table 1: Models with small contrast (above) and large contrast (below).

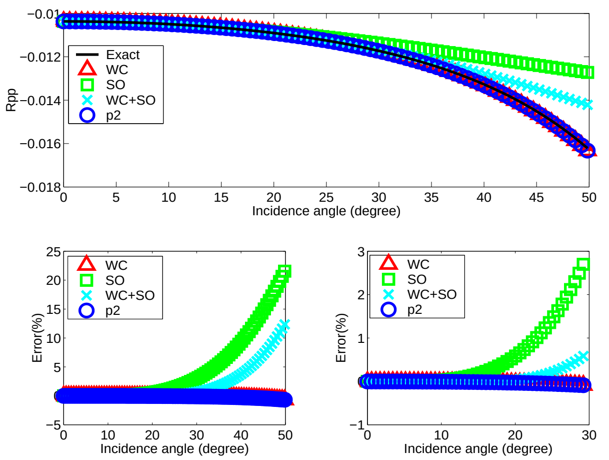


**Figure 1:** Comparison of weak contrast (WC), small offset (SO), weak-contrast and small-offset (WC+SO) and pseudo-$p^2$ $(p2)$ approximations for the small contrast model in Table 1.

Figures 1 and 2 show the performance of the four approximations compared with the exact expression for $R_{PP}$ (Zoeppritz). From these Figures we can observe that for small contrast and for small incidence angles (up to 30$^o$) the approximations give better results. We also note that the weak contrast approximation presents the best results.

In order to establish an inversion process we come back to expressions (1) - (4) and consider the two assumptions together: small contrast and small incidence angle. Therefore, those expressions can be rewriten as:

$$R_{pp} \approx G_0 + G_1 \sin^2\theta. \quad (5)$$

Each one of the expressions (1) - (4) have to be treated separately to become an expression like (5).

For expression (1) we use the assumptions that $\theta$ is small and make the substitution $\tan^2\theta \approx \sin^2\theta$. Then formula (1) turns to be as expression (5) with (Shuey, 1985)

$$G_0 = \frac{1}{2} \quad \frac{\Delta\alpha}{\alpha} + \frac{\Delta\rho}{\rho} \quad (6)$$

and

$$G_1 = \frac{1}{2} \quad \frac{\Delta\alpha}{\alpha} - 4\frac{\beta^2}{\alpha^2} \quad \frac{\Delta\rho}{\rho} + 2\frac{\Delta\beta}{\beta} \quad . \quad (7)$$

Regarding expression (3), we use again the fact of a small incidence angle, and discard the $p^4$-term. Then it becomes the same expression defined by equations (5) - (7).

The small contrast assumption permit us to do not consider the quadratic term with respect to the elastic contrast in expression (4) and then it becomes exactly as formula (1).

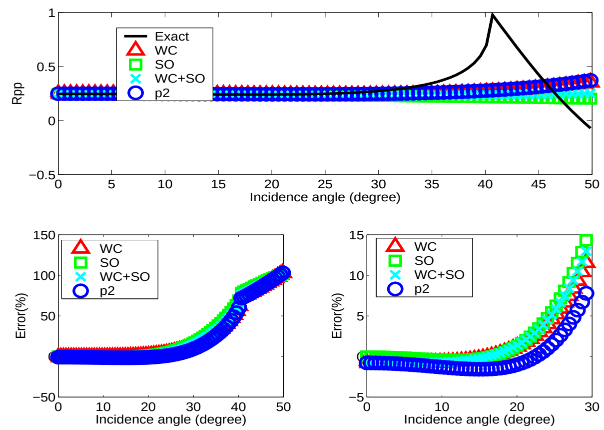


**Figure 2:** Comparison of weak contrast (WC), small offset(SO), weak-contrast and small-offset (WC+SO) and pseudo-$p^2$ $(p2)$ approximations for the large contrast model in Table 1.

The approximation defined by expression (2) has the shape of expression (5) already. Although the parameters $G_0$ and $G_1$ are not the same as equation (6) and (7).

The reflection coefficient is then approximated by expressions (5) - (7), using a least-square procedure we can obtain the values of the intercept $G_0$ and the gradient $G_1$. Table 2 shows the accuracy (the percentual error of each parameter) of this inversion process when we add a $q\%$ white noise to the exact reflection coefficient's curve of the models in Table 1.

| $q$ | Error $G_0$ | Error $G_1$ |
|---|---|---|
| 0 | 0.31 | 29.15 |
| 5 | 0.56 | 30.37 |
| 10 | 1.16 | 34.13 |
| 15 | 1.42 | 48.12 |
| 30 | 2.95 | 82.31 |

| $q$ | Error $G_0$ | Error $G_1$ |
|---|---|---|
| 0 | 1.12 | 21.35 |
| 5 | 1.16 | 21.53 |
| 10 | 1.53 | 20.80 |
| 15 | 2.02 | 22.68 |
| 30 | 2.52 | 23.55 |

Table 2: Percentual error for $G_0$ and $G_1$ when a $q\%$ white noise is added on the exact $R_{PP}$ curve to the small contrast model (left) and the large contrast model (right).

Note that only the intercept gives reliable results. Although, we have got a good approximation for this parameter it is not very useful because its formula does not have an explicit relation between the elastic parameters, or their contrasts. That is, we can only have an ideia of the sum of the contrast in the P-wave velocity and the density. When we use the impedance concept we can establish better relations to recuperate the elastic parameter, or their contrasts.

## Impedance-Type Approximation

As seen by the recent literature (Connolly, 1999; Santos & Tygel, 2004) the impedance concept seems to be a good choice to approach the reflection coefficient. The ideia is to find an impedance function $I = I(\alpha,\ \beta,\ \rho,\ \theta)$, for which the reflection coefficient can be given, or approximated, by an expression similar to the one of normal-incidence case, i.e.,

$$R_{PP} = \frac{I(\alpha_2,\ \beta_2,\ \rho_2,\ \theta_2) - I(\alpha_1,\ \beta_1,\ \rho_1,\ \theta_1)}{I(\alpha_2,\ \beta_2,\ \rho_2,\ \theta_2) + I(\alpha_1,\ \beta_1,\ \rho_1,\ \theta_1)} = \frac{I_2 - I_1}{I_2 + I_1} \quad (8)$$

The elastic impedance function (EI) proposed by Connoly (1999) is given by

$$I = N_0 \rho^{1-4K\sin^2\theta} \alpha^{\sec^2\theta} \beta^{-8K\sin^2\theta}, \quad (9)$$

where $N_0$ is a normalization constant and $K = \beta^2/\alpha^2$. This result assumes that $K$ is constant and $\theta_1 = \theta_2 = \theta$, where $\theta_1$ and $\theta_2$ are the incidence and transmission angle, respectively.

Another impedance function was proposed by Santos et al. (2002), and it is called reflection impedance function. This function is determinated considering the ray parameter $p$ ($p = \sin\theta/\alpha_1$) constant and a functional dependence between $\beta$ and $\rho$, via $\rho = b\beta^\gamma$, where $b$ is some constant of proportionality and $\gamma$ is a constant,

$$I = M_0 \frac{\rho\alpha}{\sqrt{1-\alpha^2 p^2}} \exp\{-2[2+\gamma]\beta^2 p^2\}, \quad (10)$$

with $M_0$ being a normalization constant.

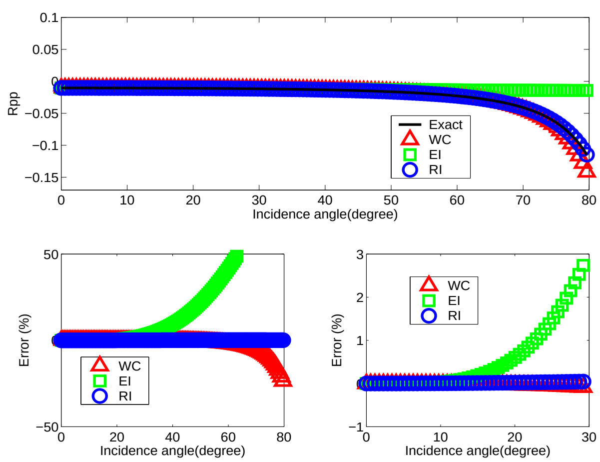


**Figure 3:**Comparison of the elastic impedance (EI), the reflection impedance (RI) and the weak contrast (WC) approximation for the small contrast model in Table 1.

As we can see from Figures 1 and 2 the weak contrast (WC) approximation is more accurate than the others, even for the quadratic approximation. Therefore, to analyze the performance of the impedance formulas, we compare them with the exact expression and the WC approximation. Figures 3 and 4 show the results. Note that, for incidence angles close to the critical one, only the reflection impedance follows the exact curve. In both models we observe that the RI approximation actually gives better results than the others.

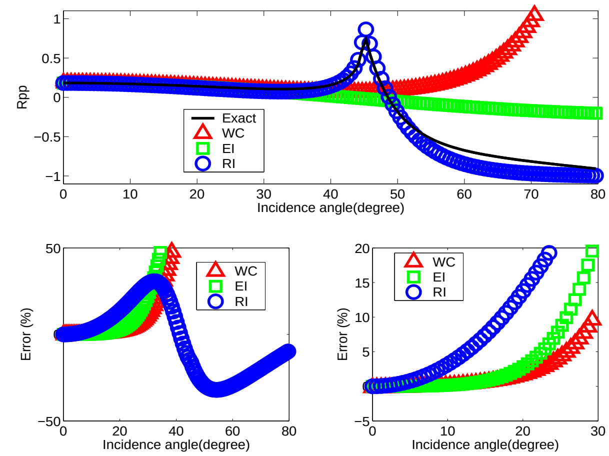


**Figure 4:**Comparison of the elastic impedance (EI), the reflection impedance (RI) and the weak contrast (WC) approximation for the large contrast model in Table 1.

From the impedance approximation (8), we define a new quantity $F = I_2/I_1$, where $I$ is one of the impedance functions given by equations (9) and (10). The reflection coefficient can then be recast as

$$R_{PP} = \frac{I_2 - I_1}{I_2 + I_1} = \frac{F-1}{F+1} \quad (11)$$

from wich

$$F = \frac{R_{PP}+1}{R_{PP}-1}. \quad (12)$$

Considering the elastic impedance function (9) and using the approximation $\tan^2\theta \approx \sin^2\theta$, we arrive at

$$\begin{aligned} \ln F &= \ln\frac{\rho_2\alpha_2}{\rho_1\alpha_1} + \ln\left[\frac{\alpha_2}{\alpha_1}\quad\frac{\rho_2\beta_2^2}{\rho_1\beta_1^2}^{-4K}\right]\sin^2\theta \\ &= A_1 + A_2\sin^2\theta \end{aligned} \quad (13)$$

Appplication of a linear least-square procedure finds the best $A_1$ and $A_2$ that fits $\ln F$ and $\sin^2\theta$.

From the reflection impedance function (10), we have

$$F = \frac{\rho_2\alpha_2}{\rho_1\alpha_1}\frac{\cos\theta}{\sqrt{1-(\frac{\alpha_2}{\alpha_1})^2\sin^2\theta}}\exp\{-2[2+\gamma][\frac{\beta_2^2-\beta_1^2}{\alpha_1^2}]\sin^2\theta\}, \quad (14)$$

which can be written as

$$F = B_1\frac{\cos\theta}{\sqrt{1-B_2^2\sin^2\theta}}\exp\{B_3\sin^2\theta\}. \quad (15)$$

Again we can find $B_1, B_2$ and $B_3$ in a least-square sense. However it is not possible to linearize the expression to apply linear least-square; a nonlinear solver must be used.

To analyze the impedance-type approximations we consider the 25 elastic isotropic models of Castagna & Smith

(1994). For each model we have applied the inversion processes to the case of shale over gas sand. For the elastic impedance we have used incidence angles up to $30^o$ and for the reflection impedance we use angles up to $70^o$. Figures 5 and 6 show the results. Observe that, even using larger incidence angles, the inverted parameters for the reflection impedance approximation gives smaller errors than the inverted paremeters for the elastic impedance approximation. Another feature of the RI approximation is that the three inverted parameters have a more explicit contrast relation. For example, parameter $B_2$ gives the ratio for the P-wave velocity.

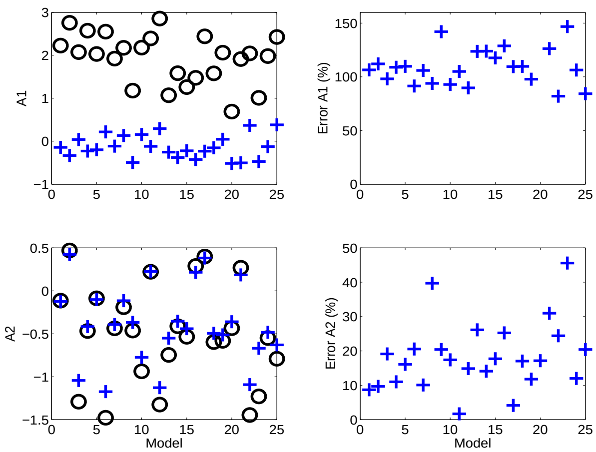


**Figure 5:**Comparison of modeled (o) and inverted (+) parameters $A_1$ (above), $A_2$ (below) for the $25$ models (shale/gas sand), considering incidence angles up to $30^o$.

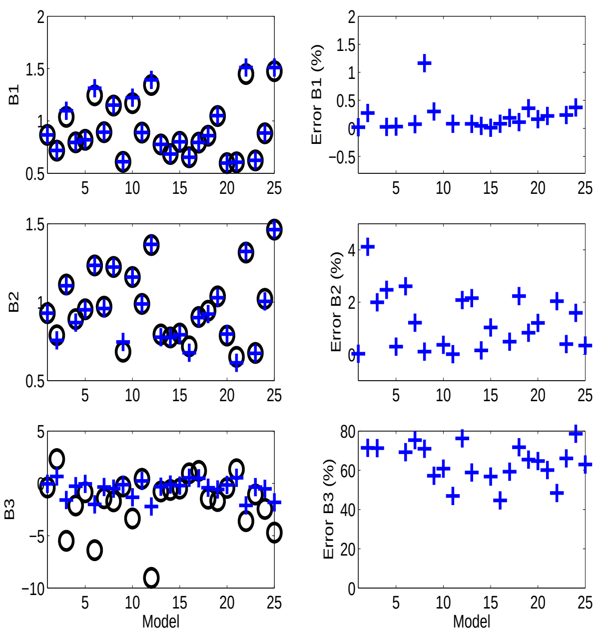


**Figure 6:**Comparison of modeled (o) and inverted (+) parameters $B_1$ (above), $B_2$ (midle) and $B_3$ (bellow) for $25$ models (shale/gas sand), considering incidence angles up to $70^o$.

## Conclusions

We have presented two kind of approximations for the Zoeppritz equations: Taylor- and Impedance-types. To discuss their accuracy we have used small and large contrast models for which the approximations curves have been plotted and compared.

Since the impedance-type approximation has provided better results, we have discussed inversion process based on this concept. Using 25 elastic models (shale/gas sand) we verified that the impedance inversion, especially the one based on the RI function, was more efficient to recuperate the elastic parameters.

Current research are being done to improve the inversion process by the reflection impedance function.

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