Uniform Resampling Using the Sinc Function

Ana Carolina Camargo and Lúcio Tunes Santos, DMA – IMECC – UNICAMP, Brazil

This paper was prepared for presentation at the 9th International Congress of The Brazilian Geophysical Society held in Salvador, Brazil, 11 – 14 September 2005.

Abstract

It is common to find problems in handling data that falls on a nonequally spaced grid. Then, we say that is necessary making an uniform resampling, i.e., interpolating the nonuniform samples of a sign in a set of equally spaced points.

In this work, it is first shown that the resampling problem can be formulated as a problem of solving a set of linear equations $Ax = b$, where x and b are vectors of the uniform and nonuniform samples, respectively, and A is a matrix of the sinc interpolation coefficients [Rosenfeld (1998)]. The solution for this system is given by the pseudoinverse matrix which is computed using singular value decomposition (SVD) in a process that is called Uniform ReSampling (URS). In large problems, the computation of the pseudoinverse is impractical.

Using the fact that the contribution of the $b(i)$'s in the computation of the $x(j)$'s, when they are distant, is very small, Rosenfeld created an algorithm that was called Block Uniform ReSampling (BURS). Such algorithm uses only a limited number of points around $t(j)$, point of the uniform grid, to calculate each uniform sample $x(j)$, decomposing thus the problem into solving a small set of linear equations for each uniform grid point. These equations are a subset of the original equations $Ax = b$ and are once again solved using SVD. The final result is both optimal and computationally efficient. A result is presented to illustrate.

Introduction

It is common to find problems in handling data that falls on a nonequally spaced grid. Then, we say that is necessary making an uniform resampling. In the most of times, this problem occurs because of the algorithms that are based in the discrete Fourier transform (DFT), require that the samples be over a Cartesian grid.

In the next section, it is shown that the resampling problem can be formulated as a problem of solving a set of linear equations $Ax = b$, and in the other sections the Uniform ReSampling and (URS) the Block Uniform ReSampling (BURS) algorithms [Rosenfeld (1998)] are presented. The last one is both optimal and efficient. The results are shown to be of excellent quality.

Problem Formulation

Let us consider a continuous real function, sampled in a finite set of nonequally spaced points, $\{\tau_1, \tau_2, \ldots, \tau_m\}$. The uniform resampling consists in to find an approximation to the function in an uniform spaced set of points, i.e., to approximate $f(t_j)$, $t_j = t_0 + j\Delta t$, $t_0 \in I\!R$, $j \in N = \{1, 2, \ldots, n\}$.

Such problem can be solved using Shannon's theorem [Oppenheim & Schafer (1989)],

Theorem *Let f be a band-limited real function, i.e., its Fourier transform is zero above some cut frequency Ω. If $\Delta t < \frac{1}{2\Omega}$, then for any $t_0 \in I\!R$*

$$f(\tau) = \sum_{j \in Z\!\!\!Z} f(t_0 + j\Delta t)\,\textit{sinc}\,\left(\frac{\tau - j\Delta t - t_0}{\Delta t}\right), \qquad (1)$$

where

$$\textit{sinc}(x) = \frac{\sin(\pi x)}{\pi x}. \qquad (2)$$

Formula (1) is readily extend to higher dimensions by replacing the sum by a multiple sum and the sinc function by a product of sinc functions.

Using Shannon's theorem, for each τ_i, $i \in M = \{1, 2, \ldots, m\}$, we can approximate

$$f(\tau_i) \approx \sum_{j=1}^{n} f(t_j)\ \text{sinc}\left(\frac{\tau_i - t_j}{\Delta t}\right), \qquad (3)$$

The set of equations above form a system of linear equations

$$Ax = b, \qquad (4)$$

where the elements of the matrix $A \in I\!R^{m\times n}$, of the vector $x \in I\!R^n$ and of the vector $b \in I\!R^m$ are given, respectively, by $a_{ij} = \text{sinc}((\tau_i - t_j)/\Delta t)$, $x_j = f(t_j)$ and $b_i = f(\tau_i)$, $i \in M$, $j \in N$. Thus, our problem is one of solving a set of m linear equations with n unknowns, Eq. (4), to determine x.

In general, the matrix A is not square. Therefore, this system must be solved in the least square sense, as we shall see in the next section.

Uniform ReSampling (URS) Algorithm

The simplest solution to the equation (4) is given by

$$x = A^{+}b, \tag{5}$$

where A^{+} denote the $n \times m$ (Moore-Penrose) pseudoinverse of A [see, e.g., Trefethen & Bau (1997)]. The pseudoinverse A^{+} provides the optimal solution to the equation $Ax = b$ in the minimal-norm least-square sense.

Computation of the pseudoinverse A^{+} is performed using singular value decomposition (SVD), which is standard component of most mathematical software packages.

Although Eq. (5) is an optimal solution, it has two inherent problems: first, the computation of A^{+} becomes impractical when the dimensions of A are too large. In the one-dimensional case, when m e n are on the order of several hundreds, inversion is practical. Second, each uniform sample, say x_j, is calculated by multiplying the jth row of A^{+} by the vector b, i.e., m multiplications (and $m-1$ additions) are involved. Using the fact that measurements that are distant from the point t_j will have coefficients with small magnitude, it was introduced an algorithm that includes only a limited number of terms in this computation. In the following section, this algorithm is desenvolved [Rosenfeld (1998)].

Block Uniform ReSampling (BURS) Algorithm

The new algorithm finds a solution of the form $x = \mathcal{A}^{+}b$, for the Equation (4), where each row of the matrix $\mathcal{A}^{+}$ contain mostly zeroes and only a restricted number of coefficients, concentrated in the neighborhood of t_j, are nonzero.

This is achieved of the following form: instead of considering all the m nonuniform points, we just consider the points $\tau_i' s$ within a radius δ from t_j, resulting a set of $\bar{m}$ points, that will be used in the computation of the x_j. Similarly, we select all Cartesian grid points within a radius Δ from t_j, with $\Delta \geq 1.5\delta$, resulting a set of $\bar{n}$ points, which will be estimated [Rosenfeld (1998)]. As it was made before, using the participating measurements we obtain the matrix $\bar{A} \in I\!R^{\bar{m}\times\bar{n}}$, which is a submatrix of A.

After that we compute the matrix $\bar{A}^{+}$, which is the $\bar{n} \times \bar{m}$ pseudoinverse matrix of $\bar{A}$. Then we isolate the row of $\bar{A}^{+}$ that corresponds to t_j. This row contains $\bar{m}$ elements,that are now inserted into the appropriate locations in the $\mathcal{A}^{+}$ matrix. That is, the entire jth row of the $\mathcal{A}^{+}$ matrix is set to zero, with the exception of these $\bar{m}$ coefficients, which are placed in the positions corresponding to their respective measurements (in the b vector). It is made for each point t_j. The result is an $n \times m$ matrix $\mathcal{A}^{+}$, which contains mostly zeroes, except for a narrow band along its "diagonal."

The BURS algorithm is very efficient because it changes the problem of computing the pseudoinverse of a large matrix by computing n pseudoinverse matrixes shorter. Moreover, the majority of the elements of the matrix $\mathcal{A}^{+}$ is zero and the solution is given by $x = \mathcal{A}^{+}b$, which has easy computation.

In the next sections, two examples are presented to show the quality of the methods previously described.

Numerical Experiments I

As a first one-dimensional example, we took the function $f(x) = 10\sin(x)\cos(10x) + \cos(3x) - \sin(2x)$ sampled onto $m = 256$ nonuniform points. The image was reconstructed onto $n = 128$ point Cartesian grid using URS and BURS algorithms. As it was expected, the results were satisfactory. The URS and BURS algorithms are shown in Figure 1 and 2 respectively. In the URS algorithm, it was calculated the pseudoinverse of an $(m \times n)$ matrix while in the BURS algorithm it were calculated 128 pseudoinverses which sizes were from 5×5 to 17×9.

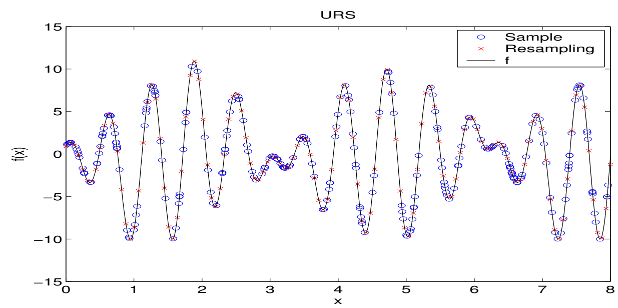

Figure 1: Uniform resampling of the function $f(x) = 10\sin(x)\cos(10x) + \cos(3x) - \sin(2x)$ using the URS algorithm.

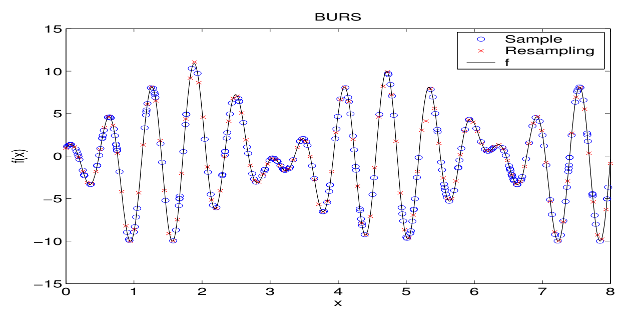

Figure 2: Uniform resampling of the function $f(x) = 10\sin(x)\cos(10x) + \cos(3x) - \sin(2x)$ using the BURS algorithm , $\delta = 0.17$ and $\Delta = 0.30$.

After that, we took the same function, but now, sampled onto $m = 4096$ nonuniform points. This function was resampled onto $n = 2048$ equally spaced points using BURS. It were computed 2048 pseudoinverses, which sizes were from 23×18 to 64×35 and the result can be seen in Figure 3. The URS algorithm was impractical computationally for this problem.

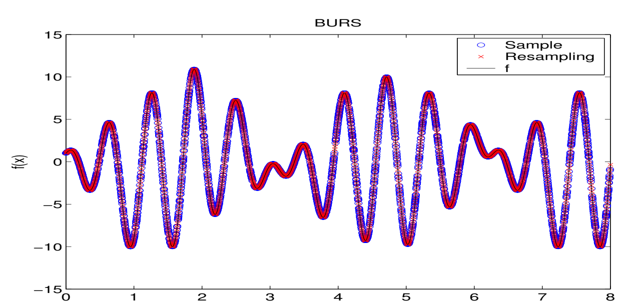

Figure 3: Uniform resampling of the function $f(x) = 10\sin(x)\cos(10x) + \cos(3x) - \sin(2x)$ using the BURS algorithm, $\delta = 0.04$ and $\Delta = 0.07$.

The last example shows the efficiency of the BURS algorithm in large problems.

Numerical Experiments II

To show the potential of the interpolation scheme for seismic purposes, we depict in Figure 4 a single interface homogeneous acoustic model and the respectives rays and traces for one shot with receivers at non-equally spaced receiver locations. Both algorithms, URS and BURS were applied to simulate the corresponding seismic section for equally spaced receiver locations. Figure 5 shows the results, were we can observe a better performance for the BURS algorithm.

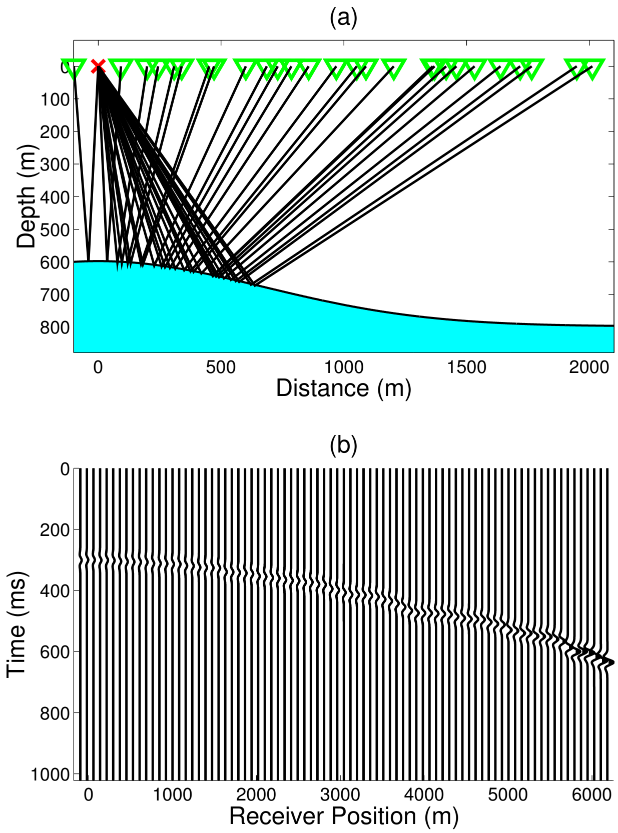

Figure 4: (a) Acoustic model and rays for a shot with receivers at non-equally spaced points. (b) Corresponding seismic section.

Conclusions

It was seen a new gridding algorithm that is both optimal and efficient. The original problem of resampling over a uniform grid was first formulated as a problem of solving a set of linear equations. These solution is obtained using the pseudoinverse. This method, the URS algorithm, is optimal in the minimal-norm least-squares sense. The BURS algorithm is a suboptimal counterpart of the URS method, which is efficient and practical. Only a limited number of measurements are used to generate each uniform grid point. An appropriate set of linear equations is constructed and subsequently solved using SVD.

The new method was applied in resampling of seismograms. From a seismic section, which the receivers were not equally spaced, it were used URS and BURS algorithms for simulate a seismic section with equally spaced recievers. It was shown that the BURS algorithm gave better results.

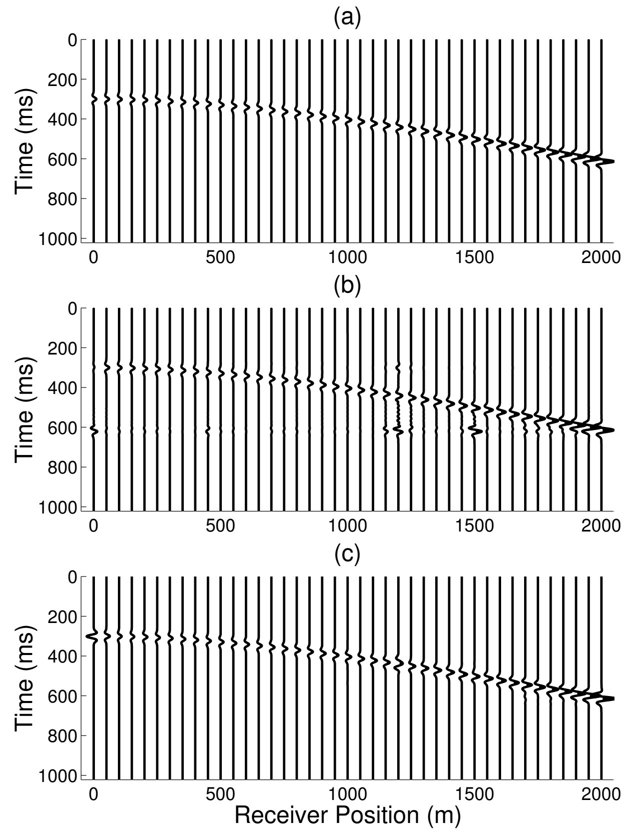

Figure 5: (a) Modeled seismic section for equally spaced receiver locations. (b) Interpolated section using URS. (c) Interpolated section using BURS, $\delta = 300$ and $\Delta = 550$.

Acknowledgements

We thank CNPq (Grant 30716512/2003-5) & FAPESP (Grants 01/01068-0 and 03/09838-5), Brazil, and the sponsors of the WIT – Wave Inversion Technology Consortium, Germany.

References

Oppenheim, A. V. & Schafer, R. W. , 1989, Discrete–Time Signal Processing, Prentice–Hall.

Rosenfeld, D. , **40**, 14–23, 1998, An Optimal and Efficient New Gridding Algorithm Using Singular Value Decomposition, Magnetic Ressonance in Medicine.

Trefethen, L. N. & Bau, D. , 1997, Numerical Linear Algebra, SIAM.