# Image-wave remigration in elliptically anisotropic media

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## Abstract

The image-wave equations for the problems of depth and time remigration in elliptically isotropic media are second-order partial differential equations similar to the acoustic wave equation. The propagation variable is the vertical velocity or the medium ellipticity rather than time. In this work, we derive these differential equations from the kinematic properties of anisotropic remigration. The objective is to enable the construction of subsurface images that correspond to different degrees of medium anisotropy. In this way, "anisotropy panels" can be obtained in a completely analogous way to velocity panels for a migration velocity analysis.

## Introduction

When a seismic migration is repeatedly carried out using different velocity models, the images of the seismic reflectors are positioned at different depth locations. To transform these migrated reflector images from one to another in a direct way, i.e., without going back to the original time section, is a seismic imaging task that can be achieved by a remigration, also known as velocity continuation or residual or cascaded migration (Rothman et al., 1985; Larner and Beasley, 1987; Fomel, 1994). Is is not difficult to accept that when migrating several times with slightly different velocity models, the sequence of images of a certain reflector creates an impression of a propagating wavefront. This "propagating wavefront" was termed an "image wave" by Hubral et al. (1996). The propagation variable, however, is not time as is the case for conventional physical waves, but the migration velocity. Moreover, due to the different kinematic behaviour, this image-wave propagation is not described by a conventional (acoustic or elastic) wave equation.

The kinematic behaviour of image waves as a function of the (constant) migration velocity has been studied in time (Fomel, 1994; Hubral et al., 1996; Fomel, 2003a,b) and in depth (Hubral et al., 1996; Schleicher et al., 2004). In this paper, we extend the idea of image waves to the remigration of images as a function of the medium anisotropy. For simplicity, we study the situation in media with elliptical isotropy, which can be described with one additional medium parameter. We choose the parameter describing the medium ellipticity to be the ratio between the squares of the vertical and horizontal velocities. We investigate the variation (or "propagation") of the reflector image as a function of this parameter.

## Derivation of the image-wave equation

The variation of the position of a reflector image when the medium anisotropy changes is to become the kinematics of the image-wave propagation of the image wave as a function of the medium anisotropy. Therefore, we study the behaviour of a single point on the image of a seismic reflector when the medium ellipticity varies. This situation can be understood in analogy to the propagation of a Huygens wave emanating from a secondary source.

The derivation starts by the construction of the "Huygens image wave", i.e.,, the set of points that describe the location of the reflector point after a variation of the propagation variable. Next, the coordinates of the original image point are replace by derivatives, in this way constructing an image eikonal equation the solution of which is the Huygens image wave. In a last step, the simplest of all second-order partial differential equations that generate this image eikonal equation is identified as the image-wave equation.

**Elliptically anisotropic medium.** An elliptically anisotropic medium is characterized by possessing a vertical symmetry axis and identical properties in all horizontal directions. Such a medium is described by four independent elastic parameters.

**Propagation velocity.** For seismic imaging purposes, the most important medium parameter is the velocity of seismic wave propagation. Here, we need expressions for this parameter for an quasi-P wave in elliptically anisotropic media. In a homogeneous elliptically anisotropic medium, the propagation of a quasi-P wave takes place in a plane (Helbig, 1983). For simplicity, we assume this plane to be the $(x, z)$-plane. Therefore, we can treat the problem as a two-dimensional one. All formulas below can readily be extended to 3D by adding corresponding $y$ components. Within the $(x, z)$-plane, the modulus of the group velocity can be written in dependence on the propagation direction according to

$$v(\theta) = \left[\frac{\sin^2\theta}{A_{11}} + \frac{\cos^2\theta}{A_{33}}\right]^{-1/2}, \qquad (1)$$

where $A_{11}$ and $A_{33}$ are diagonal elements of the density-normalized elastic tensor. Moreover, $\theta$ is the angle between the propagation direction and the vertical $z$-axis.

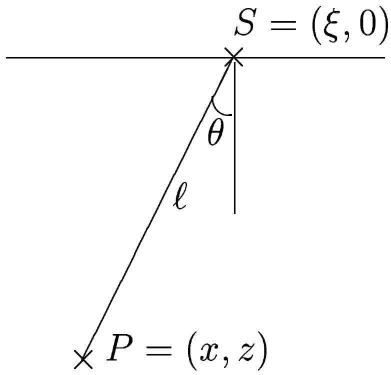


Figure 1: Zero-offset ray from source $S=(\xi,0)$ to a single reflector point $P=(x,z)$.

As a consequence of the medium anisotropy, the propagation velocities of the quasi-P wave depend on the propagation direction. In particular, there are different wave velocities in the vertical and horizontal directions. From equation (1), we recognize that the vertical ($\theta=0$) and horizontal ($\theta=\pi/2$) velocities are given by

$$v=\sqrt{A_{33}} \quad \text{and} \quad u=\sqrt{A_{11}},$$

respectively.

**Zero-offset configuration.** We assume that the migrated section to be remigrated has been obtained from zero-offset (or stacked) data under application of a zero-offset migration. The coincident source-receiver pairs where localized at a planar horizontal surface ($z=0$) at points $S=(\xi,0)$ (see Figure 1).

We denote by $x$ and $z$ the coordinates of a certain point $P$ within the medium under consideration. Moreover, we denote by $\ell$ its distance from a source $S$, such that $\ell^2=(x-\xi)^2+z^2$. The propagation angle of a wave that propagates from $S=(\xi,0)$ to $P=(x,z)$ thus satisfies $\cos\theta=z/\ell$ and $\sin\theta=(x-\xi)/\ell$. Therefore, we find the following alternative representation for the modulus of the group velocity vector in explicit dependence of the coordinates of point $P$ rather than the propagation angle $\theta$,

$$\begin{aligned} v(x,z) &= \ell\left[\frac{(x-\xi)^2}{A_{11}}+\frac{z^2}{A_{33}}\right]^{-1/2} \\ &= \ell v\left[\varphi(x-\xi)^2+z^2\right]^{-1/2}. \end{aligned} \tag{2}$$

Here, we have introduced the medium ellipticity $\varphi=A_{33}/A_{11}=v^2/u^2$.

**Traveltime.** With these results on the propagation velocity, we are now ready to describe the traveltime $T$ of a wave that was emitted and registered at $S$ and reflected at $P$. From formula (2) for the propagation velocity as a function of the coordinates of $P$, we obtain the desired traveltime as

$$T(\xi;x,z)=\frac{2\ell}{v(x,z)}=\frac{2}{v}\left[\varphi(x-\xi)^2+z^2\right]^{1/2}, \tag{3}$$

where the factor 2 is due to the symmetry of equation (1), i.e., $v(\theta)=v(\theta+\pi)$.

## Remigration

Seismic remigration tries to establish a relationship between two media of wave propagation in such a way that identical seismic surveys on their respective surfaces would yield the same seismic data. One of these media is the wrong velocity model used for the original migration. The other medium represents the updated model within which a new image of the subsurface needs to be constructed.

### *Variation of vertical velocity*

Let us suppose that the original migration has been realized with a model $M_0$ with same ellipticity $\varphi$ as used in the updated model $M$, but a different vertical velocity $v_0$. In this old model, the same diffraction traveltime $T$ of equation (3) is consumed by a different wave, reflected at a different point $P_0=(x_0,z_0)$. It is therefore given by equation (3) upon substitution of $x$, $z$, and $v$ by $x_0$, $z_0$, and $v_0$.

**Huygens image wave.** To derive the desired image-wave equation, we follow the lines of Hubral et al. (1996). Firstly, we need to find the set of all points $P=(x,z)$ in medium $M$ for which the diffraction traveltime of equation (3) is equal to the corresponding diffraction traveltime of point $P_0=(x_0,z_0)$ in medium $M_0$. In other words, we are interested in localizing the so-called Huygens wave for this kind of image-wave propagation. This Huygens image wave then describes the position $z(x)$ of the image at the "instant" $v$ that "originated" at the "instant" $v_0$ at point $P_0$. For this purpose, we equal the times $T$ of $P$ and $P_0$, resulting in

$$F(x,z,\xi,v)=\frac{\varphi}{v^2}(x-\xi)^2+z^2-\frac{\varphi}{v_0^2}(x_0-\xi)^2-z_0^2=0. \tag{4}$$

This equation represents a family of curves $z(x;\xi)$ that, for a fixed $\xi$, connect all points $P$ in model $M$ that possess the same diffraction traveltime $T(\xi;x,z)$ as $P_0$ in model $M_0$ for the same $\xi$.

The set of points $P$ such that $T(\xi;x,z)$ is equal to $T(\xi;x_0,z_0)$ for *all* values of $x$ and $z$ is given by the envelope of this family of curves described by $F(x,z,\xi,v)$. This envelope is the mentioned Huygens image wave that represents the image in model $M$ of point $P_0$ in model $M_0$. Application of the envelope condition $\partial F/\partial\xi=0$ to equation (4) yields the stationary value

$$\xi=\frac{v^2x_0-v_0^2x}{v^2-v_0^2}, \tag{5}$$

which, when substituted back in equation (4) leads to

$$z=\frac{v}{v_0}\sqrt{z_0^2-\varphi v_0^2\frac{(x-x_0)^2}{v^2-v_0^2}}. \tag{6}$$

Equation (6) describes the position of the Huygens image wave for depth remigration that was excited with the initial conditions $(x_0,z_0;v_0)$. For an isotropic medium, where

$\varphi = 1$, the above expression reduces to the one derived by Fomel (1994) or Hubral et al. (1996).

The corresponding position of the Huygens image wave for time remigration can be obtained from equation (6) by converting depth to vertical time according to $z = v\tau/2$ and $z_0 = v_0\tau_0/2$. The time domain version of equation (6) reads then

$$\tau = \sqrt{\tau_0^2 - 4\varphi\frac{(x-x_0)^2}{v^2 - v_0^2}}. \tag{7}$$

**Eikonal equation.** The Huygens image wave of equation (6) describes the variation of a single point $P_0$ on a reflector image under variation of the vertical velocity $v$, starting at an initial velocity $v_0$. To transform this expression into one that describes the variation of any arbitrarily shaped reflector image for arbitrary velocity variations, we need to eliminate these initial conditions from equation (6). In other words, we need to replace the constants $x_0$, $z_0$, and $v_0$ in equation (6) by derivatives, so as to describe image-wave propagation for any set of initial conditions.

For this purpose, we introduce the image-wave eikonal $v = \mathcal{V}(x,z)$. An explicit expression for $\mathcal{V}(x,z)$ can be found by solving equation (6) for $v$. By replacing $v$ by $\mathcal{V}(x,z)$ in equation (6), taking the derivatives with respect to $x$ and $z$ of the resulting expression, and using them to eliminate the constants $x_0$, $z_0$, and $v_0$ from equation (6), we find the following differential equation for $\mathcal{V}$,

$$\mathcal{V}_x^2 + \varphi\mathcal{V}_z^2 - \frac{\varphi\mathcal{V}}{z}\mathcal{V}_z = 0. \tag{8}$$

Its solution for initial conditions $(x_0, z_0; v_0)$ is equation (6) solved for $v$. This differential equation (8) is the image-wave eikonal equation for depth remigration in elliptically anisotropic media. It describes the kinematics of image-wave propagation for any arbitrary set of initial conditions as a function of the vertical velocity.

The corresponding procedure applied to equation (7) yields the image-wave eikonal equation for time remigration,

$$\mathcal{V}_x^2 - \frac{4\varphi}{\tau v}\mathcal{V}_\tau = 0, \tag{9}$$

where now $\mathcal{V} = \mathcal{V}(x,\tau)$.

**Image-wave equation.** Now we want to find a partial differential equation such that equation (8) is its associated eikonal equation. In other words, upon substitution of the ray-theory ansatz $p(x,z,v) = p_0(x,z)f[v - \mathcal{V}(x,z)]$ into our desired differential equation, the leading-order terms need to provide equation (8). From the leading-order terms of the second derivatives of this expression, we recognize that the second-order partial differential equation

$$p_{xx} + \varphi p_{zz} + \frac{\varphi v}{z}p_{vz} = 0 \tag{10}$$

is the simplest one to fulfill this condition. Any additional terms involving arbitrary combinations of $p$ and its first derivatives with respect to $x$, $z$, or $v$, do not alter the associated eikonal equation. Therefore, we refer to equation (10) as the image-wave equation for depth remigration in elliptically anisotropic media under variation of the vertical velocity.

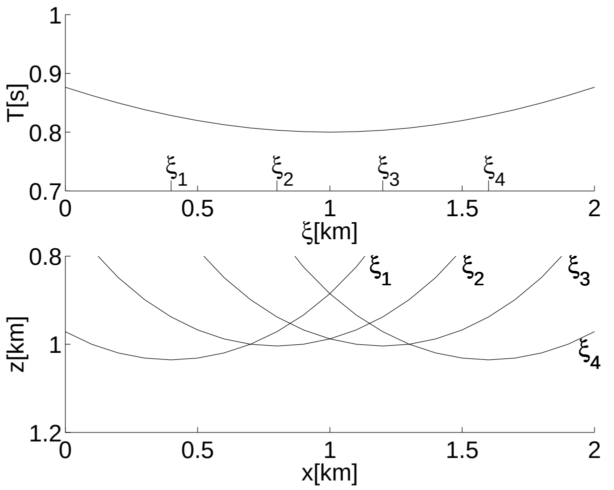


Figure 2: Top: Diffraction traveltime for a point $P_0$ with coordinates $x_0 = 1$ km, $z_0 = 1$ km in model $M_0$ with $\varphi_0 = 0.2$ and $v = 2.5$ km/s. Bottom: Family of isochrons for this point $P_0$, calculated in a model $M$ with $\varphi = 0.8$, at the four points $\xi_1 = 0.4$ km, $\xi_2 = 0.8$ km, $\xi_3 = 1.2$ km, and $\xi_4 = 1.6$ km.

Correspondingly, equation (9) leads to an image-wave equation for time remigration,

$$p_{xx} - \frac{4\varphi}{v\tau}p_{v\tau} = 0. \tag{11}$$

It is to be observed that a change of variables $\omega = v/\sqrt{\varphi}$ transforms this equation into

$$p_{xx} + \frac{4}{\omega\tau}p_{\omega\tau} = 0, \tag{12}$$

which is the corresponding equation in isotropic media (Fomel, 1994; Hubral et al., 1996). Thus, time remigration under variation of the vertical velocity in elliptically anisotropic media can be realized by the same computational program as in isotropic media.

### *Variation of medium ellipticity*

In elliptically anisotropic media, a remigration can be realized upon the variation of a second parameter, the medium ellipticity. Therefore, we now suppose that the original migration has been realized with a model $M_0$ with same vertical velocity $v$ as used in the updated model $M$, but a different ellipticity $\varphi_0$. As before, the same diffraction traveltime $T$ of equation (3) corresponds to a $P_0 = (x_0, z_0)$ in the old model and a set of points $P = (x, z)$ in the new model. The top part of Figure 2 shows the diffraction traveltime for a set of parameters $(x_0, z_0, \varphi_0, v)$.

**Huygens image wave.** Again, to derive the desired image-wave equation, we need to find the the Huygens wave for

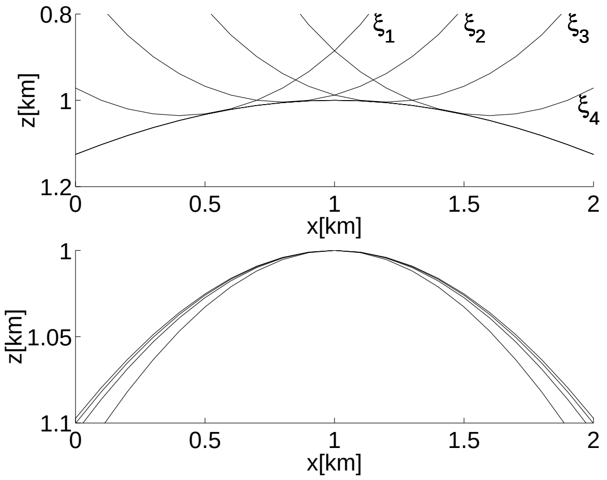


Figure 3: Top: The family of isochrons of Figure 2 together with the Huygens image wave of equation (15). Bottom: Different Huygens image waves for the same point $P_0$, indicating the propagation of the Huygens image wave. The curves are depicted for $\varphi = 0.8$, 2.0, 4.0, and 10.

this kind of image-wave propagation. This Huygens image wave then describes the position $z(x)$ of the image at the "instant" $\varphi$ that "originated" at the "instant" $\varphi_0$ at point $P_0$. Equaling the times $T$ of $P$ and $P_0$, we find

$$F(x,z,\xi,\varphi) = \varphi(x-\xi)^2 + z^2 - \varphi_0(x_0-\xi)^2 - z_0^2 = 0. \quad (13)$$

This equation represents a family of curves $z(x;\xi)$ that, for a fixed $\xi$, connect all points $P$ in model $M$ that possess the same diffraction traveltime $T(\xi;x,z)$ as $P_0$ in model $M_0$ for the same $\xi$. The bottom part of Figure 2 depicts four of these curves as obtained from equation (13) for four different values of $\xi$.

The set of points $P$ such that $T(\xi;x,z)$ is equal to $T(\xi;x_0,z_0)$ for *all* values of $x$ and $z$ is given by the envelope of this family of curves described by $F(x,z,\xi,\varphi)$. This envelope is the mentioned Huygens image wave that represents the image in model $M$ of point $P_0$ in model $M_0$. Application of the envelope condition $\partial F/\partial\xi = 0$ to equation (13) yields the stationary value

$$\xi = \frac{\varphi x - \varphi_0 x_0}{\varphi - \varphi_0}, \quad (14)$$

which, when substituted back in equation (13) leads to

$$z = \sqrt{z_0^2 + \varphi\varphi_0 \frac{(x-x_0)^2}{\varphi - \varphi_0}}. \quad (15)$$

Equation (15) describes the position of the Huygens image wave that was excited with the initial conditions $(x_0, z_0; \varphi_0)$. In the top part of Figure 3, this Huygens image wave is added to the four isochrons of the bottom part of Figure 2. This nicely demonstrates the characteristic property of the Huygens image wave (15), i.e., being the envelope of the set of isochrons described by equation (13). The bottom part of Figure 3 depicts a set of these Huygens image waves for different values of the medium ellipticity $\varphi$.

As for the velocity variation, the substitution $z = v\tau/2$ and $z_0 = v\tau_0/2$ transfers the Huygens image wave to the time-migrated domain, resulting in

$$\tau = \sqrt{\tau_0^2 + \frac{4\varphi\varphi_0}{v^2}\frac{(x-x_0)^2}{\varphi-\varphi_0}}. \quad (16)$$

**Eikonal equation.** The Huygens image wave of equation (15) describes the variation of a single point $P_0$ on a reflector image under variation of the medium ellipticity $\varphi$, starting at an initial ellipticity $\varphi_0$. As before, we need to eliminate these initial conditions from equation (15), i.e., we need to replace the constants $x_0$, $z_0$, and $\varphi_0$ in equation (15) by derivatives.

As the next step, we introduce the corresponding image-wave eikonal $\varphi = \Phi(x,z)$. As before, an explicit expression for $\Phi(x,z)$ can be found by solving equation (15) for $\varphi$. By replacing $\varphi$ by $\Phi(x,z)$ in equation (15), taking the derivatives with respect to $x$ and $z$ of the resulting expression, and using them to eliminate the constants $x_0$, $z_0$, and $\varphi_0$ from equation (15), we find the image-wave eikonal equation for $\Phi$,

$$\Phi_x^2 - \frac{2\Phi^2}{z}\Phi_z = 0. \quad (17)$$

Its solution for initial conditions $(x_0, z_0; \varphi_0)$ is equation (15) solved for $\varphi$. The image-wave eikonal equation describes the kinematics of image-wave propagation for depth remigration in elliptically anisotropic media for any arbitrary set of initial conditions as a function of the medium ellipticity.

The same procedure applied to equation (16) yields the corresponding image-wave eikonal equation for time remigration,

$$\Phi_x^2 + \frac{8\Phi^2}{\tau v^2}\Phi_\tau = 0, \quad (18)$$

where now $\Phi = \Phi(x,\tau)$.

**Image-wave equation.** Again, the last step is to find a partial differential equation such that equation (17) is its associated eikonal equation. In other words, upon substitution of the ray-theory ansatz $p(x,z,\varphi) = p_0(x,z)f[\varphi - \Phi(x,z)]$ into our desired differential equation, the leading-order terms need to provide equation (17). From the leading-order terms of the second derivatives of this expression, we recognize that the second-order partial differential equation

$$p_{xx} + \frac{2\varphi^2}{z}p_{z\varphi} = 0 \quad (19)$$

is the simplest one to fulfill this condition. Therefore, we refer to equation (19) as the image-wave equation for depth remigration in elliptically anisotropic media under variation of the ellipticity.

It is important to observe that the image-wave equation (19) can be transformed into a partial differential equation with constant coefficients. Upon the introduction of the new variables $\gamma = 1/\varphi$, and $\zeta = z^2/4$, the image-wave equation (19) takes the form

$$p_{xx} - p_{\gamma\zeta} = 0. \qquad (20)$$

This transformation is meaningful from an implementational point of view, since for differential equations with constant coefficients, it is generally much easier to find stable FD implementations.

As a final word on the image-wave equation (19) or its constant-coefficient version, let us mention that both equations do not depend on the vertical velocity $v$ but only on the medium ellipticity $\varphi$. Thus, it can be expected that depth image-wave remigration in elliptically anisotropic media should be relatively insensitive to the actual value of the vertical velocity. This, in turn, points towards a potentially broad applicability of the image-wave concept for elliptically anisotropic remigration even in inhomogeneous media.

Correspondingly, equation (18) leads to an image-wave equation for time remigration,

$$p_{xx} - \frac{8\varphi^2}{\tau v^2} p_{\varphi\tau} = 0. \qquad (21)$$

It is to be observed that the same change of variables as before, $\omega = v/\sqrt{\varphi}$, now with varying $\varphi$, also transforms this equation into the corresponding equation (12) for isotropic media. In other words, also time remigration under variation of the medium ellipticity can be realized by the same computational program as in isotropic media.

In fact, a careful analysis of time remigration under a simultaneous variation of both, vertical velocity $v$ and medium ellipticity $\varphi$ shows that even in this situation, the final image-wave equation can be transformed into equation (12) that depends on the above combined parameter $\omega$ only. By substitution of the definition of the medium ellipticity $\varphi$ into the above expression for the transformed variable $\omega$, we observe that $\omega = v/\sqrt{v^2/u^2} = u$ is nothing else than the horizontal velocity. In other words, time remigration in elliptically anisotropic media is independent of the vertical velocity and depends only on the variation of the horizontal velocity.

## Conclusions

In this work, we have derived a set of second-order partial differential equations that work as image-wave equations for remigration in elliptically anisotropic media. They describe the propagation of a reflector image in time and depth remigration as a function of the vertical velocity and the medium ellipticity. To this end, we have studied the kinematics of the image wave in such media to derive the corresponding eikonal equations.

The description of the position of the reflector image as a function of the medium ellipticity can be very useful for the detection of this parameter. A set of migrated images for different medium ellipticities can be obtained from a single migrated image without the need for multiple anisotropic migrations. From additional information on the correct reflector position or a focusing analysis, the best fitting value of the medium ellipticity can then be determined.

The probably most interesting application of this procedure would start with an initial condition of an isotropic medium, described by unit ellipticity, i.e., $\varphi_0 = 1$. Since isotropic migration is a very well understood field, the image-wave equation could then be used to transform an isotropically migrated image, which can be obtained with one of the highly sophisticated migration methods that are nowadays available, into an image that corresponds to an elliptically anisotropic medium.

In the case of time remigration, the image-wave equation shows that the position of the reflector image in elliptically anisotropic media depends on the horizontal velocity only. This implies that a migration velocity analysis based on time migration can only detect this parameter. In particular, this means that there is no way to distinguish an elliptically anisotropic medium from an isotropic one on the basis of time migration only.

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