True Amplitude One-Way Waves

Gabriela Melo, Jörg Schleicher & Amélia Novais, State University of Campinas

This paper was prepared for presentation at the 9th International Congress of The Brazilian Geophysical Society held in Salvador, Brazil, 11-14 September 2005.

Abstract

In homogeneous media, the two-way wave operator (or just wave operator) can be substituted by the product of two one-way wave operators that form the standard factoring of the wave equation. These one-way operators generate two one-way wave equations, which are fast tools for modeling and migration since they allow us to separate the two-way wave (or full wave), which is the solution of the full wave equation, into two one-way waves, a downgoing and an upgoing one. Since these one-way waves are solutions of the wave equation in homogeneous media, they satisfy the same approximate (ray-theory) differential equations, eikonal and transport equations, as does the full wave, that is, their traveltime and amplitudes agree in first order approximation with those of the full wave. Since these one-way wave equations produce correct traveltimes even in inhomogeneous media, they have been used in wave equation migration WEM (Claerbout, 1971; 1985). However, in this case, they do not correctly treat reflection amplitudes. Therefore, using them, only kinematically correct migrated images are obtained. In this paper, we study how the one-way wave operators need to be changed to make the one-way waves produce the same amplitudes of the full wave and keep the traveltime agreement.

Introduction

The one-way wave operators used in the standard factoring of the wave equation allow us to separate solutions of the wave equation (waves that propagate in both downgoing and upgoing directions) into downgoing and upgoing waves (one-way waves). In a homogeneous medium, the traveltimes and amplitudes of the one-way waves are the same as the full wave, since they satisfy the same differential equations. These equations are the eikonal equation that governs the traveltime, and the transport equation that describes the amplitudes. If we must handle the wave equation in an inhomogeneous medium, the one-way wave amplitudes become different from those of the full wave, while the traveltimes remain the same. Recently, Zhang et al. (2003) have shown how to modify the one-way wave operators in order to make modified one-way waves with the same amplitudes and traveltime as the full wave.

First, the domain is changed trough the Fourier transform in time and transverse spatial coordinates (x, y). Thus, the wave equation, which is a partial differential equation, becomes an ordinary differential equation. Here, depth (z) is the direction of propagation. Then, the standard factoring of the wave equation for homogeneous medium, where the velocity of propagation is constant, is modified for the case when velocity depends on direction of propagation, that is, $c = c(z)$. Finally, the factoration is generalized to fully heterogeneous medium where $c = c(x, y, z)$.

The implementation of the modified one-way operators in the case where $c = c(x, y, z)$ involves applying the square-root of a differential operator as a pseudo-differential operator. This square-root operator can be represented as an integral in which a rational function of the transverse Laplacian appears in the integrand. Through this one can do an asymptotic analysis of the implementation results of the modified one-way waves.

These modified one-way waves were recently presented by Zhang et al (2003), who also show that the solutions of these waves have traveltime and amplitudes that are asymptotically consistent with the solution of the corresponding full wave equation. And, in this sense, the one-way waves are called true amplitude one-way waves.

Using this true amplitude one-way waves, a true amplitude WEM can be formulated, starting from to classical WEM of Claerbout(1971;1985). This modified WEM produces a reflector map, and amplitudes that agree with the solutions of the full wave equation.

Modified one-way waves

In the following discussion, we describe how to obtain the true amplitudes one-way waves for the cases in which the velocity of propagation c of the medium are $c = constant$, $c = c(z)$, $c = c(x, y, z)$.

Let us introduce the definition of Fourier transform, and his inverse form, that is used in this work

$$F(k_x, k_y, z, \omega) = \int dx\, dy\, dt\, F(x, y, z, t) \times \exp\{i(-\omega t + k_x x + k_y y)\}, \tag{1}$$

$$F(x, y, z, t) = \frac{1}{(2\pi)^3} \int dk_x\, dk_y\, d\omega\, F(k_x, k_y, z, \omega) \times \exp\{i(\omega t - k_x x - k_y y)\}. \tag{2}$$

The study is done around the tridimensional wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = 0. \tag{3}$$

Despite the wave equation has an exact solution for some cases, in an inhomogeneous medium it generally has not. Therefore, we take the ray theoretic aproximation (or WKBJ solution)

$$u = A \exp\{-i\omega\varphi(x, y, z)\}, \tag{4}$$

as a candidate to the solution of the wave equation. Here A is the amplitude of the wave and φ its traveltime.

For constant propagation velocity, it is possible to write the wave equation in the frequency/wave vector domain as

$$\begin{aligned}\mathcal{L}u &= 2\frac{\partial^2 u}{\partial z^2} + \left(\frac{\omega^2}{c^2} - (k_x^2 + k_y^2)\right)u + k_z^2 u \\ &= \left[\frac{\partial}{\partial z} \pm ik_z\right]\left[\frac{\partial}{\partial z} \mp ik_z\right]u, \qquad (5)\end{aligned}$$

where

$$k_z = sign(\omega)\sqrt{\frac{\omega^2}{c^2} - \vec{k}^2} = \frac{\omega}{c}\sqrt{1 - \frac{(c\vec{k})^2}{\omega^2}}, \qquad (6)$$

$$\vec{k}^2 = k_x^2 + k_y^2, \qquad (7)$$

$$\vec{k} = (k_x, k_y). \qquad (8)$$

Here, the symbol $\mathcal{L}$ is the differential operator of the full wave equation, the vector $\vec{k}$ is the transverse wave vector, and

$$\left[\frac{\partial}{\partial z} \pm ik_z\right]\left[\frac{\partial}{\partial z} \mp ik_z\right]u, \qquad (9)$$

is the standard factoring of the wave equation.

The one-way differential operators

$$\left[\frac{\partial}{\partial z} \pm ik_z\right], \qquad (10)$$

generate the one-way wave equations

$$\left[\frac{\partial}{\partial z} \pm ik_z\right]u = \left[\frac{\partial}{\partial z} \pm ik_z\right]A^{\pm}\exp\{\mp ik_z z\} = 0, \qquad (11)$$

the solutions of which are a downgoing and an upgoing wave, respectively. Therefore, as we can see in (5), these waves are also solutions of the wave equation in a homogeneous medium. So, they satisfy the same differential equations and obviously they have the same traveltime and amplitudes as the full wave. Thus, for a homogeneous medium, the standard one-way wave equations are the true amplitude one-way wave equations that we are looking for.

The next step is to derive one-way waves for the case that the velocity depends on the direction of propagation, that is, $c = c(z)$.

Upon the introduction of the slowness vector $\vec{p}$

$$\vec{p} = \frac{\vec{k}}{\omega}, \quad p_z = \frac{k_z}{\omega} = \frac{1}{c(z)}\sqrt{1 - (c(z)\vec{p})^2}, \qquad (12)$$

the wave equation can be rewritten as follows

$$\frac{\partial^2 u}{\partial z^2} + \omega^2 p_z^2 u = 0. \qquad (13)$$

Using again the slowness vector, the wave equation solutions and its derivatives become

$$u = A(z,\vec{p})\exp\{-i\omega\varphi(z,\vec{p})\}, \qquad (14)$$

$$\frac{\partial u}{\partial z} = \frac{\partial A}{\partial z}\exp\{-i\omega\varphi\} - iA\omega\frac{\partial \varphi}{\partial z}\exp\{-i\omega\varphi\}, \qquad (15)$$

$$\begin{aligned}\frac{\partial^2 u}{\partial z^2} &= \frac{\partial^2 A}{\partial z^2}\exp\{-i\omega\varphi\} - \frac{\partial A}{\partial z}i\omega\frac{\partial \varphi}{\partial z}\exp\{-i\omega\varphi\} \\ &\quad -\frac{\partial A}{\partial z}i\omega\frac{\partial \varphi}{\partial z}\exp\{-i\omega\varphi\} - iA\omega\frac{\partial^2 \varphi}{\partial z^2}\exp\{-i\omega\varphi\} \\ &\quad -A\omega^2\left(\frac{\partial \varphi}{\partial z}\right)^2\exp\{-i\omega\varphi\}. \qquad (16)\end{aligned}$$

Substituting this solution and its derivatives in the wave equation, one obtains

$$\left\{-\omega^2\left[\left(\frac{\partial \varphi}{\partial z}\right)^2 - p_z^2\right]A - i\omega\left[2\frac{\partial \varphi}{\partial z}\frac{\partial A}{\partial z} + \frac{\partial^2 \vec{p}}{\partial z^2}A\right] + O(1)\right\}\exp\{-i\omega\varphi\} = 0. \qquad (17)$$

The equation above provides the eikonal and transport equation, respectively, in the domain of transverse slowness vector

$$\frac{\partial \varphi^{\pm}}{\partial z} = \pm p_z. \qquad (18)$$

$$\frac{\partial A^{\pm}}{\partial z} - \frac{1}{2c^3(z)p_z^2}\frac{\partial c(z)}{\partial z}A^{\pm} = 0. \qquad (19)$$

In the eikonal equation, the upper sign corresponds to the downgoing wave and the lower sign corresponds to the upgoing one. However, the transport equation is the same for both solutions since only p_z^2 appears in it.

Let us examine the differential equations for the one-way waves. Consider the one-way waves coming from the standard factoring

$$\left[\frac{\partial}{\partial z} \pm ik_z\right]A^{\pm}\exp\{-i\omega\varphi^{\pm}\}. \qquad (20)$$

Through these equations we obtain the respective eikonal and transport equations

$$\frac{\partial \varphi^{\pm}}{\partial z} = \pm p_z, \qquad (21)$$

$$\frac{\partial A^{\pm}}{\partial z} = 0. \qquad (22)$$

We can see that eikonal equation (21) for the one-way wave agrees with the eikonal equation (18) for the full wave. However, the transport equations (19) and (22) are different. Thus, while the traveltimes of both waves agree, the amplitudes do not. For the purpose of making the one-way waves amplitudes equal to the full wave amplitudes, the one-way operators must be modified. For this end, a new term is added to these operators

$$\left[\frac{\partial}{\partial z} \pm ik_z + \alpha\right]. \qquad (23)$$

Then, the modified equation (20) leads to the new differential equations

$$\frac{\partial \varphi^{\pm}}{\partial z} = \pm p_z, \qquad (24)$$

$$\frac{\partial A^{\pm}}{\partial z} = \alpha A^{\pm}. \qquad (25)$$

From a comparison of equations (19) and (25), it becomes clear that the modified one-way operators

$$\left\{\frac{\partial}{\partial z} \mp i\omega p_z - \frac{1}{2c^3(z)p_z^2}\frac{\partial c(z)}{\partial z}A\right\}u = 0, \qquad (26)$$

are kinematically and dynamically equivalent to the full wave equation. By construction, these equations have the same eikonal and transport equations as full wave, as desired.

Finally, we go to the case of the fully heterogeneous medium where $c = c(x,y,z)$. In such a situation, the Fourier transform is not going to facilitate the problem, since a convolution appears when applying the Fourier transform to the first term of the wave equation. Therefore, the above last one-way equations obtained for $c = c(z)$ are modified to include variations in all dimensions. Substitution of $c(z)$ by $c = c(x,y,z)$ in these equations yields

$$\left\{\frac{\partial}{\partial z} \pm ik_z - \frac{\omega^2}{2c^3(x,y,z)k_z^2}\frac{\partial c(x,y,z)}{\partial z}\right\}u = 0. \qquad (27)$$

To make the following calculations easier, we rewrite the expression above as

$$\left\{\frac{\partial}{\partial z} \mp ik_z - \frac{1}{2c(x,y,z)}\frac{\partial c(x,y,z)}{\partial z}[1+W]\right\}u = 0, \qquad (28)$$

where

$$W = W(x,y,z) = \frac{c^2(x,y,z)\vec{k}^2}{\omega^2 - (c(x,y,z)\vec{k})^2}. \qquad (29)$$

At this point, let us briefly comment on how the involved operators look like in the original (x,y,z,t) domain. The easiest expression is that of the operator corresponding to $(c\vec{k})^2$ which reads

$$\begin{aligned} -(c\nabla_T)^2 &= -c^2(\vec{\rho},z)\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] \\ &\quad -c(\vec{\rho},z)\left[\frac{\partial c(\vec{\rho},z)}{\partial x}\frac{\partial}{\partial x} + \frac{\partial c(\vec{\rho},z)}{\partial y}\frac{\partial}{\partial y}\right], \end{aligned} \qquad (30)$$

where

$$\vec{\rho} = (x,y). \qquad (31)$$

A little more of a problem is that the operator k_z involves the square-root of a differential operator.

We rewrite the one-way wave equations with the amplitude correction term as follows

$$\mathcal{L}_\pm u = \left[\frac{\partial}{\partial z} \pm \Lambda\right]u - \Gamma u = 0, \qquad (32)$$

where Λ e Γ are the differential operators in the original variables relatives to λ e γ in the frequency wave vector domain, λ and γ being

$$\lambda = ik_z = \frac{i\omega}{\sqrt{1 - \frac{(c(x,y,z)\vec{k})^2}{\omega^2}}}, \qquad (33)$$

$$\gamma = \frac{c_z}{2c}(1 + W(x,y,z)). \qquad (34)$$

A contour integration in the complex plane, yields an exact representation for λ in such a way that the differential operator does not appear inside a square-root as desired,

$$\lambda = ik_z = \frac{i\omega}{c}\left\{1 - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}W(x,y,z)ds\right\}, \quad \omega^2 > (c\vec{k})^2. \qquad (35)$$

Note that the expression $\omega^2 - (c\vec{k})^2$ can represent the transversal two-way wave operator, i.e.,

$$L_T \equiv L_T(s;\vec{\rho},z,t) = \frac{\partial^2}{\partial t^2} - s^2(c\nabla_T)^2. \qquad (36)$$

At this point, the operators Λ and Γ can be written symbolically as

$$\Lambda = \frac{1}{c}\frac{\partial}{\partial t}\left\{I - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}L_T^{-1}(c\nabla_T)^2 ds\right\}, \qquad (37)$$

$$\Gamma = \frac{c_z}{2c}(I + L_T^{-1}(1;\vec{\rho},z,t)(c\nabla_T)^2). \qquad (38)$$

After some rearrangements, and the use of an auxiliar function q that satisfies

$$L_T q = \left\{\frac{\partial^2}{\partial t^2} - s^2(c\nabla_T)^2\right\}q = (c\nabla_T)^2 u, \quad z > 0,\ t > 0, \qquad (39)$$

the final expression for the one-way wave operators leads

$$L_\pm u = \frac{\partial u}{\partial z} \pm \frac{1}{c}\frac{\partial u}{\partial t} \mp \frac{1}{\pi c}\frac{\partial}{\partial t}\int_{-1}^{1}\sqrt{1-s^2}\,q\,ds + \frac{c_z}{2c}[u + q(1;\vec{\rho},z,t)] = 0. \qquad (40)$$

Now, this modified one-way waves can be applied in wave equation migration, in order to derive a true-amplitude WEM. We begin by examining the classic WEM of Claerbout (1971; 1985) and then modify the equations and boundary conditions to attain the true amplitude WEM.

The one-way waves obtained by the standard factoring is used even in an inhomogeneous medium. Supose we handle a single source experiment with the reflected wavefield being observed at $z = 0$. Let D be the source wavefield and U be the observed wavefied, and supose they are solutions of

$$\left\{\begin{array}{l} \left(\frac{\partial}{\partial z} + \Lambda\right)D = 0, \\ D(x,y,z=0;\omega) = -\delta(\vec{x} - \vec{x}_s), \\ \vec{x} = (x,y,z), \\ \vec{x}_s = (x_s,y_s,0), \end{array}\right.$$

and

$$\left\{\begin{array}{l} \left(\frac{\partial}{\partial z} - \Lambda\right)U = 0, \\ U(\vec{x}_s;\omega) = Q(x,y;\omega). \end{array}\right. \qquad (41)$$

Then, standard WEM defines the *impedance* or *reflectivity* function as

$$R(x,y,z) = \frac{1}{2\pi}\int \frac{U(\vec{x};\omega)}{D(\vec{x};\omega)}d\omega. \qquad (42)$$

This imaging method produces a reflector map based on the fact that constructive interference between the phases of the two waves produces a large amplitudes exact where the reflectors are, and destructive interference produces small amplitudes where they are not. Unfortunately, this method does not return accurate amplitude information. To get true amplitude information the function R needs to be calculated differently, using the modified one-way waves instead the standard ones. Let m_D and m_U be the solutions of

$$\begin{cases} \left(\frac{\partial}{\partial z} + \Lambda - \Gamma\right) m_D(\vec{x};\omega) = 0, \\ m_D(\vec{x};\omega) = -\frac{1}{2}\Lambda^{-1}\delta(\vec{x}-\vec{x}_s), \end{cases}$$

and

$$\begin{cases} \left(\frac{\partial}{\partial z} - \Lambda - \Gamma\right) m_U = 0, \\ m_U(\vec{x}_s;\omega) = Q(x,y;\omega). \end{cases} \qquad (43)$$

Note that, beyond modifying the differential equations according to previous discussion, also the boundary conditions for the source wavefield must be modified. The reason is that the boundary value must only account for the impulsive nature of the source in transverse direction.

Using the modified waves m_D and m_U function R must bo calculated from

$$R(x,z,z) = \frac{1}{2\pi}\int \frac{m_U(\vec{x};\omega)}{m_D(\vec{x};\omega)}d\omega. \qquad (44)$$

To demonstrate that the modified one-way waves constructed here have the same leading order amplitudes as the full wave, we will present some numerical examples.

Conclusions

The WEM in inhomogeneous media produces incorrect amplitude information when realized using the standard one-way wave equations. The true amplitude one-way waves introduced by Zhang et al. (2003), produce WEM with correct migrated amplitudes, and can be also used in forward modeling. To give meaning to the new one-way wave operators, pseudo-differential operator theory is necessary. The true amplitude one-way waves equations provide solutions which traveltime and amplitudes agree with the solution of the full wave equation in ray-theoretical approximation. The new one-way waves can be used to construct a true amplitude WEM, by modifying the standard process of WEM.

References

[1] Claerbout, J. F., Toward a unified theory of reflector imaging: Geophysics, 36, 3, 467-481, 1971.

[2] Claerbout, J. F., Imaging the Earth's Interior: Blackwell Scientific Publications, Inc, 1985.

[3] Zhang, Y., Zhang, G. e Bleistein, N., Theory of true amplitude one-way wave equations and true amplitude common-shot migration: 8th Ann. Internat. Mtg., Brazil. Geophys. Soc., Expanded Abstracts, SBGf-292:1–4, 2003.