# Polarization of plane waves in viscoelastic anisotropic media

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## Abstract

Homogeneous and inhomogeneous time-harmonic plane waves propagating in unbounded viscoelastic anisotropic media are generally elliptically polarized. Exceptions are P and S waves propagating along some specific directions, along which they are linearly polarized. A typical case are SH waves propagating in a plane of symmetry of a viscoelastic anisotropic medium. Two most important characteristics of the polarization are the orientation of the axes of the polarization ellipse and its eccentricity. They both usually vary considerably with the direction of wavefront propagation, and with varying strength of inhomogeneity of the considered plane wave. The orientation of the polarization ellipse generally differs from the direction of wavefront propagation, being usually closer to the direction of the energy flux. The eccentricity of the polarization ellipse depends particularly strongly on the inhomogeneity of the plane wave. For homogeneous plane waves, the polarization is usually nearly linear, with large eccentricity. The eccentricity decreases with increasing inhomogeneity of the wave. For strongly inhomogeneous plane waves, the polarization ellipse becomes nearly circular, eccentricity being very small. The eccentricity of the polarization ellipse usually also decreases in a vicinity of singular directions.

## Introduction

We investigate the polarization of homogeneous and inhomogeneous time-harmonic plane waves propagating in an unbounded viscoelastic anisotropic medium in an arbitrarily specified direction. The plane wave is given by the relation

$$u_j(x_k,t) = U_j \exp[-\mathrm{i}\omega(t - p_n x_n)] \ , \qquad (1)$$

where $x_k$ are Cartesian coordinates, $u_j, p_j$ and $U_j$ are Cartesian components of the complex-valued displacement vector $\mathbf{u}$, slowness vector $\mathbf{p}$ and displacement amplitude vector $\mathbf{U}$, respectively. The vector $\mathbf{U}$ is called here the *polarization vector*. It can be arbitrarily normalized. Futher, $t$ is time and $\omega$ is a fixed, positive real-valued circular frequency. Equation (1) represents a plane wave if, and only if, $U_j$ and $p_j$ are chosen in such a way that (1) satisfies the equation of motion. This requirement yields a system of linear equations for $U_1$, $U_2$ and $U_3$:

$$a_{ijkl} p_j p_l U_k = U_i \ , \quad i = 1,2,3 \ . \qquad (2)$$

Here $a_{ijkl}$ are complex-valued, frequency-dependent, density-normalized viscoelastic moduli. They satisfy the symmetry relations $a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij}$. The condition of solvability of the system of equations (2) reads

$$\det[a_{ijkl} p_j p_l - \delta_{ik}] = 0 \ . \qquad (3)$$

The constraint relation (3) can be used to determine the slowness vector $\mathbf{p}$. When the relevant complex-valued slowness vector $\mathbf{p}$ is determined, it can be used in Equation (2) to calculate the polarization vector $\mathbf{U}$.

## Determination of the polarization vector $\mathbf{U}$

The basic problem in studying polarization of the plane wave (1) in viscoelastic anisotropic media is not the solution of Equation (2) for $\mathbf{U}$, but the determination of the slowness vector $\mathbf{p}$, which satisfies (3). Commonly, the slowness vector is specified in the form $\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}$, where $\mathbf{P}$ is the real-valued propagation vector (perpendicular to the wavefront), oriented in the direction of the propagation of the wavefront, and $\mathbf{A}$ is the real-valued attenuation vector (perpendicular to the plane of constant amplitude), oriented in the direction of the maximum decay of amplitudes. Usually, the slowness vector is specified by the directions of propagation and attenuation vectors, see, e.g., Aki and Richards (1980), Krebes and Le (1994) or Carcione and Cavallini (1995). Such a specification of the slowness vector leads, however, to certain complications, see Červený and Pšenčík (2005a,b). Several alternative parameterizations have been proposed by Červený (2004). Here we use the so-caled *mixed specification*:

$$\mathbf{p} = \sigma\mathbf{n} + \mathrm{i}D\mathbf{m}\ . \tag{4}$$

In (4), $\mathbf{n}$ and $\mathbf{m}$ are real-valued, mutually perpendicular unit vectors. It is obvious that the vector $\mathbf{n}$ is parallel to the propagation vector $\mathbf{P}$ so that $\mathbf{n}$ is also perpendicular to the wavefront. The two vectors, $\mathbf{n}$ and $\mathbf{m}$ specify the so-called *propagation-attenuation plane* $\Sigma^{\|}$. The vectors $\mathbf{P}$ and $\mathbf{A}$ are situated in it. The scalar *inhomogeneity parameter* $D$, $-\infty < D < \infty$, is a measure of the inhomogeneity of the plane wave, $|D|$ specifying the *inhomogeneity strength*. The plane wave is called homogeneous for $D = 0$. In this case the vectors $\mathbf{P}$ and $\mathbf{A}$ are parallel. The plane wave is called inhomogeneous if $D \neq 0$. In this case, the vectors $\mathbf{P}$ and $\mathbf{A}$ make a non-zero angle, called the *attenuation angle*. The vectors $\mathbf{n}$ and $\mathbf{m}$, and the inhomogeneity parameter $D$ represent uniquely the parameters of the plane wave under consideration, and may be chosen arbitrarily.

The complex-valued scalar quantity $\sigma$ in (4) is a quantity to be sought. Algebraic equation for $\sigma$ can be obtained by inserting (4) into (3):

$$\det[a_{ijkl}(\sigma n_j + p_j^{\Sigma})(\sigma n_l + p_l^{\Sigma}) - \delta_{ik}] = 0\ . \tag{5}$$

Eq.(5) is of the sixth degree with complex-valued coefficients. Its six roots correspond to three plane-wave modes, $P$, $S1$ and $S2$, propagating along and against $\mathbf{n}$. In special cases, particularly in planes of symmetry, Equation (5) can be factorized to equations of the fourth and second degree, and their solutions can be sought analytically.

Once Equation (5) is solved for $\sigma$, the relevant slowness vector can be determined from (4), and the system (2) can be solved for $\mathbf{U}$. System (2) must be supplemented by a suitable normalization condition for $\mathbf{U}$. In this contribution, $\mathbf{U}$ is normalized so that $\mathbf{U}\mathbf{U}^*$=1. Here $\mathbf{U}^*$ denotes the complex conjugate of $\mathbf{U}$.

## Numerical examples. Polarization diagrams

Here we investigate numerically the polarization of homogeneous and inhomogeneous plane P, S1 and S2 waves propagating in an unbounded viscoelastic anisotropic medium. Three different models are used: a) Model MJ of Jakobsen et al. (2003) of a viscoelastic medium of hexagonal symmetry; b) Model MJ ELAST, derived from the model MJ by specifying imaginary parts of viscoelastic moduli zero; c) Model MJ ROT generated by rotating the model MJ. We use the following colour convention: polarization diagrams of the fastest wave (in examples shown, it is always the P wave) are red, of the intermediate wave (the S1 wave) are black, and of the slowest wave (the S2 wave) are blue.

**Model MJ** We consider plane waves propagating in a medium of hexagonal symmetry whose complex-valued moduli were obtained by Jakobsen et al. (2003). We choose the model corresponding to the frequency of approximately 35 Hz and consider the density of 1000 kg/m$^3$. The $6 \times 6$ matrix $\mathbf{A}$ of complex-valued, density-normalized viscoelastic moduli of hexagonal symmetry with vertical axis of symmetry, measured in (km/s)$^2$ reads:

$$\mathbf{A} = \mathbf{A_1} - \mathrm{i}\mathbf{A_2}, \tag{6}$$

where the matrix $\mathbf{A_1}$ reads

$$\begin{pmatrix} 46.63 & 5.98 & 4.28 & 0. & 0. & 0. \\ & 46.63 & 4.28 & 0. & 0. & 0. \\ & & 19.93 & 0. & 0. & 0. \\ & & & 13.44 & 0. & 0. \\ & & & & 13.44 & 0. \\ & & & & & 20.32 \end{pmatrix}$$

and the matrix $\mathbf{A_2}$ reads

$$\begin{pmatrix} 0.033 & 0.022 & 0.156 & 0. & 0. & 0. \\ & 0.033 & 0.156 & 0. & 0. & 0. \\ & & 1.312 & 0. & 0. & 0. \\ & & & 0.055 & 0. & 0. \\ & & & & 0.055 & 0. \\ & & & & & 0.005 \end{pmatrix} .$$

Note that the real-valued part of the matrix $\mathbf{A}$ exhibits the kiss singularity along the vertical.

We study plane waves propagating in a vertical symmetry plane, which coincides with the propagation-attenuation plane $\Sigma^{\|}$. The components of the unit vectors $\mathbf{n}$ and $\mathbf{m}$ can thus be expressed as

$$n_1 = \sin i, \quad n_2 = 0, \quad n_3 = \cos i,$$
$$m_1 = \cos i, \quad m_2 = 0, \quad m_3 = -\sin i, \tag{7}$$

where $i$ is called the *propagation angle*.

In this case, the SH plane waves are polarized linearly, in the direction perpendicular to $\Sigma^{\|}$. The polarization diagrams in the propagation-attenuation plane correspond to P and SV waves, the SH waves polarization being represented by points only. Polarization diagrams of the SV wave are usually blue because the SV wave is mostly the slowest wave. Only in some directions it becomes faster than the SH wave. Then its polarization diagrams become black.

Fig.1 shows polarization diagrams for the inhomogeneity parameter $D = 0.02$. The polarization ellipses are plotted for twenty values of the propagation angle $i$, specifying the vector $\mathbf{n}$ (perpendicular to the wavefront). For $\mathbf{n}$ pointing upwards, $i = 0^0$, for $\mathbf{n}$ pointing to the right, $i = 90^0$, see Eq.(7). The vectors $\mathbf{n}$ point from the center of the figure towards the centers of polarization ellipses. Several interesting phenomena can be observed in Fig.1.

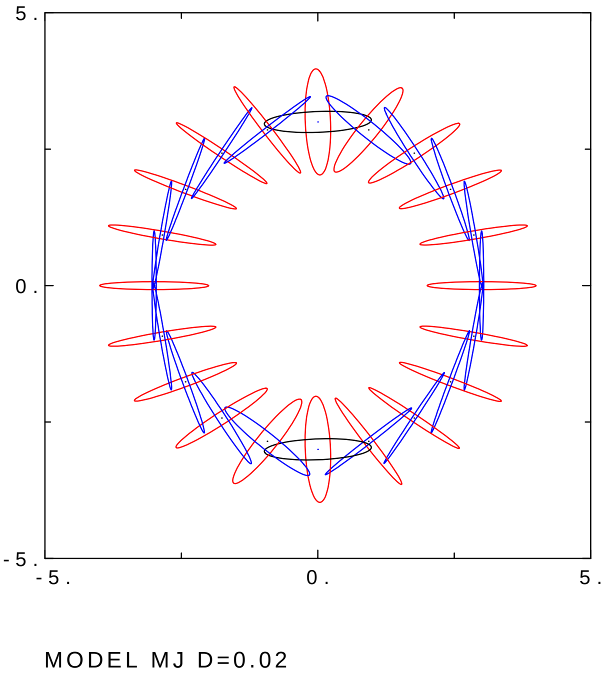


Figure 1: Polar plot of polarization diagrams of P and SV waves in a plane of symmetry of the medium (6) for $D = 0.02$. Red: the fastest wave (P wave); black: the intermediate wave; blue: the slowest wave. SH wave is polarized perpendicularly to the plane of symmetry.

a) Both P and SV waves are elliptically polarized. The longer axes of P- and SV-wave polarization ellipses are mutually perpendicular.

b) For some $\mathbf{n}$, the long axes of the polarization ellipses of P waves deviate considerably from the direction of $\mathbf{n}$. The deviations are minimum in horizontal and vertical directions. The same holds for SV waves if the deviations are measured from the direction perpendicular to $\mathbf{n}$.

c) The eccentricity of the polarization ellipses varies significantly with varying $\mathbf{n}$.

d) The eccentricities of the polarization ellipses of P and S waves for a given $\mathbf{n}$ are similar, but not the same.

e) Particularly strong changes of eccentricity can be observed close to the vertical. In the perfectly elastic case, this is the direction of the kiss singularity.

f) Anomalous behaviour of phase velocities of SV and SH waves can be observed in the vicinity of the vertical. Note that along the vertical the SV wave (black) becomes faster than the SH wave.

g) There is no symmetry in the orientation of the polarization ellipses and in their eccentricity with respect to the vertical. The symmetry exists for $D = 0$. With increasing $|D|$, asymmetry increases, see the following figures. A full symmetry can be observed for $\mathbf{n}$ and $-\mathbf{n}$.

In addition to the above observations let us add one, which is not demonstrated in Fig.1. The orientation of longer axes of the polarization ellipses of P waves is usually closer to the direction of energy flux then to the direction of $\mathbf{n}$.

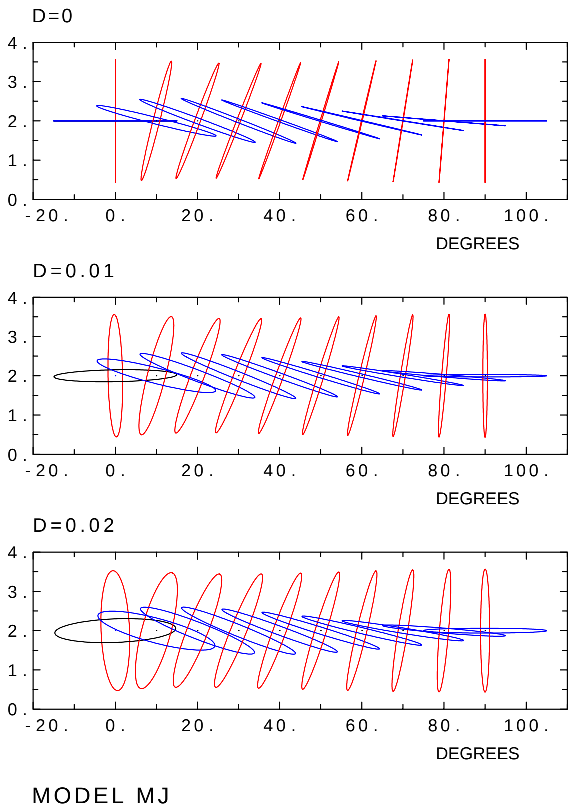


Figure 2: Polarization diagrams of P and S waves versus the propagation angle $i$ for $D = 0$ (homogeneous wave), 0.01 and 0.02. Direction of wavefront propagation is vertical. The colours as in Fig.1. Black indicates that SV wave is faster than SH wave in a given direction.

In Figs. 2 and 3, we investigate behaviour of polarization for varying values of the inhomogeneity parameter $D$, namely for $D = 0$, $0.01$, $0.02$, $0.03$, $0.05$ and $0.1$. The horizontal axis corresponds to the propagation angle $i$. The vertical direction corresponds to the direction of the vector $\mathbf{n}$, perpendicular to the wavefront. In this presentation, the deviations of the longer axes of polarization ellipses of the P wave from the direction of $\mathbf{n}$ are particularly well pronounced. Note that the longer axis is always parallel to $\mathbf{n}$ only for $i = 90^o$. For $i = 0^o$, this is true only for a homogeneous plane wave, $D = 0$. With increasing $D$, the angle $i$, for which the two directions are parallel, shifts from zero to positive values. For $D = 0.1$ in Fig.3, it is, approximately, $i = 5^o$.

As we can see from Figs. 2 and 3, the polarization ellipses have very large eccentricity for homogeneous plane waves ($D = 0$). With increasing $D$, the eccentricity decreases. For $D$ greater than those used in Fig.3, the polarization ellipses become nearly circular. Consequently, for great $D$, it becomes difficult to distinguish P and SV waves according to their polarization. Also note that for increasing $D$, the range of propagation angles $i$, for which the SV wave propagates faster than the SH wave, also increases.

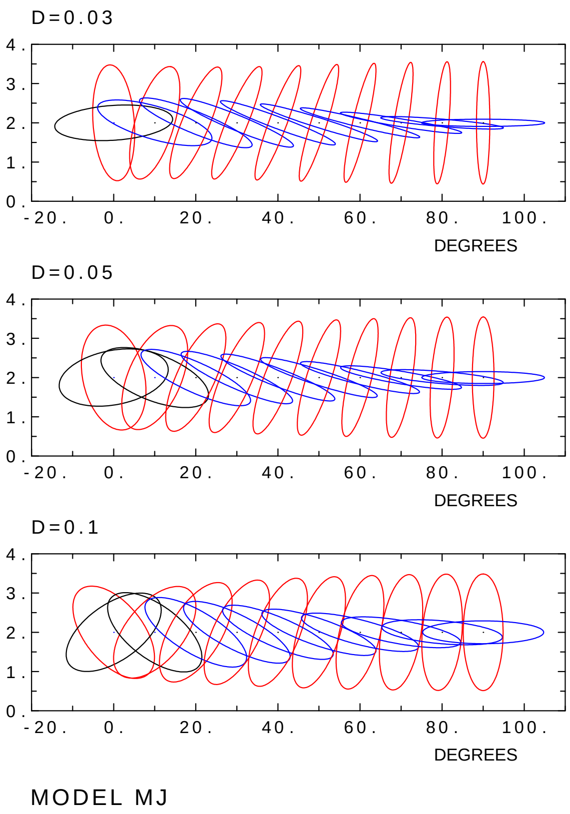


Figure 3: The same as in Fig.2 but for $D = 0.03$, $0.05$ and $0.1$.

**Model MJ ELAST** In the model MJ ELAST, the $6 \times 6$ matrix of density-normalized moduli is specified as $\mathbf{A} = \mathbf{A_1}$, see (6). Thus, we deal with a perfectly elastic anisotropic medium in this case. Let us compare the results for perfectly elastic anisotropic media with the results for viscoelastic anisotropic media, presented in the previous section.

Fig.4 shows the same as Fig.2, but for a perfectly elastic anistropic medium. For a homogeneous wave ($D = 0$), the polarization is strictly linear, both for P and SV waves. For inhomogeneous plane waves ($D \neq 0$), the polarization becomes elliptical.

Comparison of Figs.2 and 4 shows that the polarization diagrams of P and SV waves in perfectly elastic and viscoelastic anisotropic media are very similar for a given $D$ and $i$. This indicates that the perturbation methods for homogeneous as well as inhomogeneous plane waves propagating in weakly viscoelastic media, in which perfectly elastic media are used as a reference, should work very well.

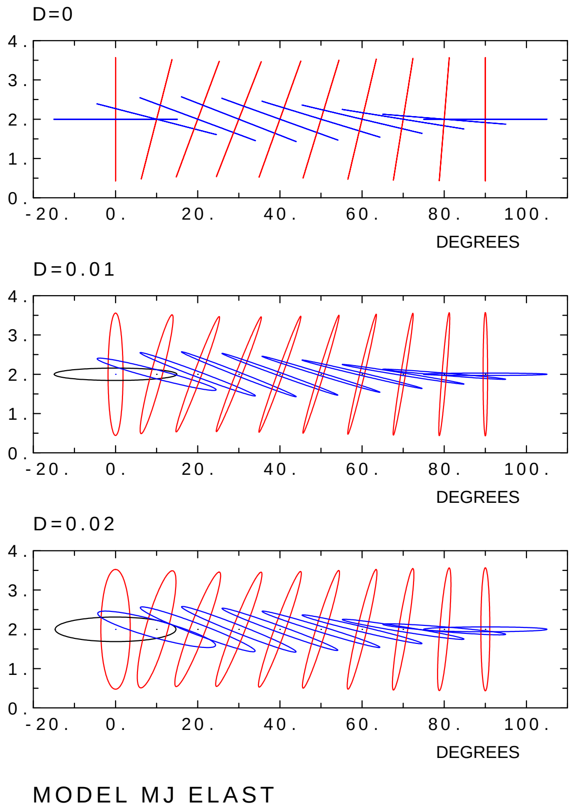


Figure 4: The same as in Fig.2 but for $\mathbf{A} = \mathbf{A_1}$ (perfect elasticity), see (6) with $\mathbf{A_2} = 0$.

**Model MJ ROT** In this section, we wish to illustrate that the algorithms described in introductory sections work safely even outside symmetry planes.

In such a case, we cannot speak about SV and SH waves any more, and thus we speak about S1 (faster) and S2 waves (slower). We use the model MJ and rotate it by 30$^o$ about the vertical ($x_3$ axis), and then by 40$^o$ about the new $x_2$ axis. Fig.5 shows projections of the polarization diagrams into the propagation-attenuation plane $\Sigma^{\parallel}$. Because $\Sigma^{\parallel}$ does not coincide with the symmetry plane, we can observe polarizations of all three plane waves. The three plots in Fig.5 correspond to $D = 0$, $0.01$ and $0.02$.

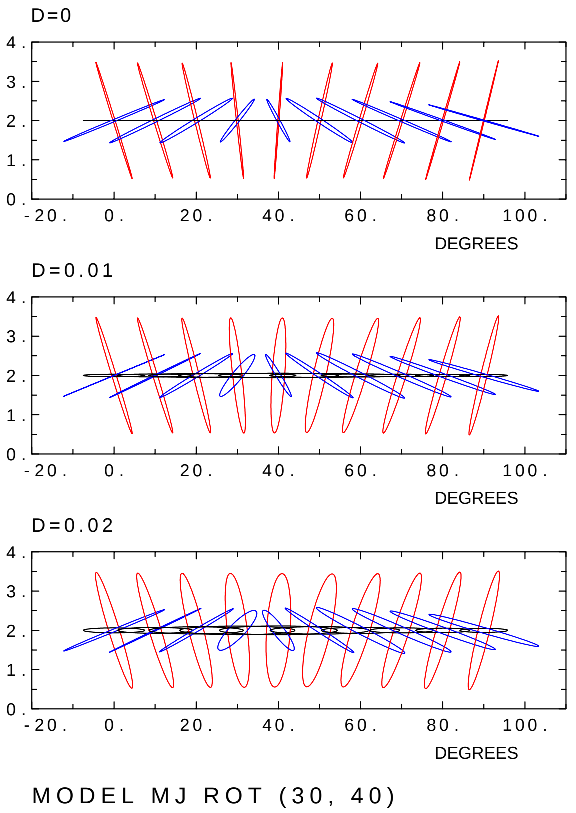


Figure 5: The same as in Fig.2 but for the rotated model (6). Black: the faster S wave.

The polarization diagrams in Fig.5 have similar features as those in Figs. 2 and 3, specifically decrease of the eccentricity with increasing $D$. Interesting is behaviour of polarization of the faster (black) S wave in Fig.5. It is polarized strictly horizontally for $D = 0$ and nearly horizontally for $D \neq 0$. The explanation of this behaviour is simple. In a medium of hexagonal symmetry, every vector is situated in a plane of symmetry. Moreover, one of the S waves is polarized linearly and its polarization is perpendicular to the plane of symmetry. Thus a slowness vector of any homogeneous plane wave is situated in a plane of symmetry and is perpendicular to the polarization of the mentioned S wave. Since the slowness vectors in Fig.5 are situated in the plane of the figure and are vertical, projection of the polarization of the considered S wave is horizontal. As $D$ increases, the polarization of this wave also becomes elliptical and starts to deviate from the horizontal.

## Acknowledgements

The authors are grateful to Morten Jakobsen for providing data for numerical examples. The research has been supported by the Consortium Project "Seismic Waves in Complex 3-D Structures", by the Research Projects 205/04/1104 and 205/05/2182 of the Grant Agency of the Czech Republic, and by the Research Project A3012309 of the Grant Agency of the Academy of Sciences of the Czech Republic.

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