Poroelastic modeling: low-frequency case

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Abstract

The classic poroelastic theory of Biot, developed in 1950's, describes the propagation of elastic waves through a porous media containing a fluid. This theory has been extensively used in various fields dealing with porous media: seismic exploration, oil/gas reservoir characterization, environmental geophysics, earthquake seismology, etc. In this work we use the Ursin formalism to derive explicit formulas for the analysis of propagation of elastic waves through a stratified 3D porous media, where the parameters of the media are characterized by piece-wise constant functions of only one spatial variable, depth. There is considered the low-frequency limit of the Biot equations.

Introduction

Poroelastic models are used in geophysics and petroleum engineering, where porous media filled with fluid and/or gas is of great interest. The best-known poroelastic theory was developed by Maurice Biot, see Biot (1956a) and Biot (1956b).

There are many works devoted to the development and application of analytical/semi-analytical methods for wave propagation analysis in stratified elastic media, see, for instance, Thomson (1950), Haskell (1953), Brekhovskih (1960), Kunetz and d'Erceville (1962), Ursin (1983), and Molotkov (1984).

The development of similar methods in the case of stratified porous media is very important too, see Allard et al. (1989), Baird et al. (1999), Molotkov (2002), and Carcione (2007).

The Ursin formalism gives a unified treatment of electromagnetic waves, acoustic waves, and the isotropic elastic waves in plane layered media. Recently, this formalism was applied to the Pride equations for simulation of the electrokinetic phenomena in layered media, see White and Zhou (2006).

In this work we apply Ursin's method for solving the Biot system in the case of the 3D poroelastic plane layered media. In the exposition of results, we follow basically to the White and Zhou work. Although the results obtained by White and Zhou allow, under certain conditions, to split Pride's equations and select only the poroelastic part, we examine the case of a more complete poroelastic system, characterized by presence in the Darcy law of an inertial force connected with the effective density of pore fluid.

Method

1. Problem. We shall consider wave propagation in a porous half-space $R = \bigcup_{k=1}^{k=N} R_k$, composed with stratified layers $R_k = \left\{ x = (x_1, x_2, x_3 \equiv z) \in \mathbb{R}^3 : z_k < z < z_{k+1} \right\}$, where $0 = z_0 < z_1 < \cdots < z_{N+1} = \infty$. Let $u = (u_1, u_2, u_3)$ and $w = (w_1, w_2, w_3)$ be the solid and relative fluid displacements, respectively. The Biot equations (low-frequency case) in the time frequency (ω) domain, at each point $x \in R$, are (time dependence of $e^{-i\omega t}$ is assumed)

$$-i\omega(\rho v + \rho_f q) = \nabla \cdot \tau + f$$
$$q = \frac{\kappa}{\eta}(-\nabla p + i\omega\rho_f v + i\omega\rho_E q + g)$$
$$-i\omega\tau = (\lambda\nabla \cdot v + C\nabla \cdot q)I + G(\nabla v + \nabla v^T) \tag{1}$$
$$i\omega p = C\nabla \cdot v + M\nabla \cdot q$$

Here: $v = -i\omega u, q = -i\omega w$ are the solid and relative fluid velocities, $f = (f_1, f_2, f_3)$ and $g = (g_1, g_2, g_3)$ the forces imposed on the solid and on the pore fluid, respectively; τ, the stress tensor; p, the pressure in the pore fluid; λ, G, the Lamé coefficients; C, M, the Biot moduli; ρ, the bulk density; ρ_f, the density of the pore fluid; ρ_E, the effective density of the pore fluid; κ, the permeability; η, the pore fluid viscosity; I, is the 3×3 identity matrix. All the material parameters are represented by piece-wise constant functions, depended only on the depth coordinate z, with the discontinuities at the points $z = z_k, k = 1, 2, \ldots, N$.

At layer boundaries we suppose that the following functions are continuous:

$$v_i, q_3, p, \tau_{i3}, i = 1, 2, 3 \tag{2}$$

The boundary conditions at the free surface $z = 0$ are

$$p = \tau_{i3} = 0, i = 1, 2, 3 \tag{3}$$

And finally, at the infinity the solution satisfies the following radiation conditions:

$$\lim_{|x| \to \infty}(v, q) = 0 \tag{4}$$

2. Ursin format. Consider the Fourier transform in the two coordinates x_1, x_2

$$\hat{X}(k_1,k_2,z) = F_{x_1x_2}(X) \equiv \iint_{R^2} e^{-i(k_1x_1+k_2x_2)} X(x_1,x_2,z)\,dx_1\,dx_2$$

Let $(k_1,k_2)^T$ be the horizontal wavenumber and $k = \sqrt{k_1^2 + k_2^2}, \gamma = k\omega^{-1}$. Applying the Fourier transform to (1) we obtain the EDO's system represented in the terms of $\hat{f}, \hat{g}, \hat{v}, \hat{q}, \hat{\tau}, \hat{p}$.

Let

$$\Omega = k^{-1} \begin{pmatrix} k_1 & k_2 & 0 \\ -k_2 & k_1 & 0 \\ 0 & 0 & k \end{pmatrix}$$

The EDO's obtained can be simplified if we define

$$\tilde{x} = \Omega x, \tilde{v} = \Omega\hat{v}, \tilde{q} = \Omega\hat{q}, \tilde{\tau} = \Omega\hat{\tau}\Omega^T, \tilde{f} = \Omega\hat{f}, \tilde{g} = \Omega\hat{g}, \tilde{p} = \hat{p}$$

A straightforward calculation uncouples this system

$$\frac{d\Phi^{(m)}}{dz} = -i\omega M^{(m)}\Phi^{(m)} + S^{(m)}, m = 1,2 \qquad (5)$$

where $\Phi^{(m)}$ are the $2n_m$ - vectors ($n_1 = 3, n_2 = 1$), defined as

$$\Phi^{(1)} = (\tilde{v}_3, \tilde{\tau}_{13}, -\tilde{q}_3, \tilde{\tau}_{33}, \tilde{v}_1, \tilde{p})^T, \Phi^{(2)} = (\tilde{v}_2, \tilde{\tau}_{23})^T$$

$S^{(m)}$ are the source $2n_m$-vectors, and $M^{(m)}$ are the $2n_m \times 2n_m$-matrices

$$M^{(m)} = \begin{pmatrix} 0 & M_1^{(m)} \\ M_2^{(m)} & 0 \end{pmatrix} \qquad (6)$$

with symmetric $n_m \times n_m$-matrices $M_1^{(m)}, M_2^{(m)}$.

For Systems 1 and 2 the submatrices and the corresponding source vectors are

$$System\,1: \quad M_1^{(1)} = \begin{pmatrix} -\beta M & \beta\gamma(C^2 - \lambda M) & -\beta C \\ \beta\gamma(C^2 - \lambda M) & \rho + \dfrac{i\omega\rho_f^2\kappa}{\eta - i\omega\rho_E\kappa} - 4\beta\gamma^2 G(C^2 - M(\lambda + G)) & 2\beta\gamma GC - \dfrac{i\omega\rho_f\gamma\kappa}{\eta - i\omega\rho_E\kappa} \\ -\beta C & 2\beta\gamma GC - \dfrac{i\omega\rho_f\gamma\kappa}{\eta - i\omega\rho_E\kappa} & -\beta(\lambda + 2G) + \dfrac{i\omega\gamma^2\kappa}{\eta - i\omega\rho_E\kappa} \end{pmatrix}$$

$$M_2^{(1)} = \begin{pmatrix} \rho & \gamma & -\rho_f \\ \gamma & G^{-1} & 0 \\ -\rho_f & 0 & \dfrac{-\eta + i\omega\rho_E\kappa}{i\omega\kappa} \end{pmatrix}, S^{(1)} = (0, -\tilde{f}_1 - \dfrac{i\omega\rho_f\kappa}{\eta - i\omega\rho_E\kappa}\tilde{g}_1, \dfrac{ik\kappa}{\eta - i\omega\rho_E\kappa}\tilde{g}_1, -\tilde{f}_3, 0, \tilde{g}_3)^T \qquad (7)$$

$$System\,2: \quad M_1^{(2)} = G^{-1}, M_2^{(2)} = \rho - G\gamma^2 + \dfrac{i\omega\rho_f^2\kappa}{\eta - i\omega\rho_E\kappa}, S^{(2)} = (0, -\tilde{f}_2 - \dfrac{i\omega\rho_f\kappa}{\eta - i\omega\rho_E\kappa}\tilde{g}_2)^T$$

where $\beta = (C^2 - M(\lambda + 2G))^{-1}$. Once $\Phi^{(1)}$ and $\Phi^{(2)}$ have been determined, we may compute

$$\tilde{q}_1 = \dfrac{\kappa}{\eta - i\omega\rho_E\kappa}(-ik\tilde{p} + i\omega\rho_f\tilde{v}_1 + \tilde{g}_1), \tilde{\tau}_{11} = \beta\{-4\gamma G(C^2 - M(\lambda + G))\tilde{v}_1 + (C^2 - \lambda M)\tilde{\tau}_{33} + 2GC\tilde{p}\}$$

$$\tilde{\tau}_{22} = \beta\{-2\gamma G(C^2 - \lambda M)\tilde{v}_1 + (C^2 - \lambda M)\tilde{\tau}_{33} + 2GC\tilde{p}\}, \tilde{q}_2 = \dfrac{i\omega\rho_f\kappa}{\eta - i\omega\rho_E\kappa}\tilde{v}_2 + \dfrac{\kappa}{\eta - i\omega\rho_E\kappa}\tilde{g}_2, \tilde{\tau}_{12} = -G\gamma\tilde{v}_2$$

$$(8)$$

3. <u>Diagonalization.</u> Let's give briefly a derivation of the diagonalization procedure. We consider matrices of the form (6), where for simplicity we drop the superscript $^{(m)}$.

Assume that $M_1 M_2$ has n distinct nonzero eigenvalues $\lambda_j^2, j = 1,2,\ldots,n$, with associated eigenvectors a_j,

$j = 1, 2, \ldots, n$, such that $a_j^T M_2 a_j = \lambda_j$. Here $\lambda_j = \sqrt{\lambda_j^2}$ with the branch chosen so that $\mathrm{Im}(\lambda_j) \geq 0$ and $\lambda_j > 0$ is real if λ_j is real. Define $b_j = \lambda_j^{-1} M_2 a_j$. This vector is an eigenvector of $M_2 M_1$ with eigenvalue λ_j^2. Using symmetricity of M_1, M_2 we obtain $a_j^T b_i = \delta_j^i$, where δ_j^i is the Kronecker delta.

Let L_1 be the $n \times n$ matrix whose j-th column is a_j, and let L_2 be the $n \times n$ matrix whose i-th column is b_i, then $L_1^{-1} = L_2^T, L_2^{-1} = L_1^T$. Introduce

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

Then $L_2 \Lambda = M_2 L_1$ and $M_1 L_2 = L_1 \Lambda$, which implies

$$M_1 = L_1 \Lambda L_1^T, \quad M_2 = L_2 \Lambda L_2^T \qquad (9)$$

Introducing the diagonal matrix $\tilde{\Lambda} = \mathrm{diag}(\Lambda, -\Lambda)$ and using (9), we finally obtain

$$M = L \tilde{\Lambda} L^{-1} \qquad (10)$$

where

$$L = \frac{1}{\sqrt{2}} \begin{pmatrix} L_1 & L_1 \\ L_2 & -L_2 \end{pmatrix}, \quad L^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} L_2^T & L_1^T \\ L_2^T & -L_1^T \end{pmatrix}$$

The explicit formulas for λ_j, a_j, b_j for Systems 1 and 2 are given in Appendix.

4. Reflection and transmission matrices. Firstly, we consider a homogeneous, source-free region of space. Dropping $^{(m)}$ we have a $2n$-dimensional system of the form (5) with M constant and $S = 0$. Let

$$\Phi = L \Psi \text{ and } \Psi = (U, D)^T \qquad (11)$$

where U, D are n-vectors, characterizing upgoing (U) and downgoing (D) waves. Then

$$\Psi(z) = \left(e^{-i\omega\Lambda(z-z_0)} U(z_0), e^{i\omega\Lambda(z-z_0)} D(z_0) \right)^T \qquad (12)$$

where z_0 is a fixed point in the same source-free region. Next consider an interface at z, where the material parameters vary discontinuously across z. We denote by $^{\pm}$ quantities evaluated at $z^{\pm} = z \pm 0$. Since Φ is continuous across z, we obtain

$$\Psi^{\pm} = J^{\pm 1} \Psi^{\mp} \qquad (13)$$

where the jump matrix is

$$J = (L^+)^{-1} L^- \equiv \begin{pmatrix} J_A & J_B \\ J_B & J_A \end{pmatrix}, J^{-1} = \begin{pmatrix} J_A^T & -J_B^T \\ -J_B^T & J_A^T \end{pmatrix}$$

and J_A, J_B are the $n \times n$-matrices

$$J_A = \frac{1}{2} \left[\left(L_2^+\right)^T L_1^- + \left(L_1^+\right)^T L_2^- \right], J_B = \frac{1}{2} \left[\left(L_2^+\right)^T L_1^- - \left(L_1^+\right)^T L_2^- \right]$$

Next, we consider a stack of layers $0 < z_1 < \ldots < z_N < \infty$. We have $\left(U_N^-, D_N^- \right)^T = J_N^{-1} \left(0, D_N^+ \right)^T$, where we have used that there is no upgoing wave below the last interface at $z = z_N$. So, we obtain

$$U_N^- = \Gamma_N D_N^-, D_N^+ = T_N D_N^- \qquad (14)$$

where

$$\Gamma_N = -\left(J_{B,N}^T\right)\left(J_{A,N}^T\right)^{-1}, T_N = \left(J_{A,N}^T\right)^{-1} \qquad (15)$$

Here Γ_N is the reflection matrix and T_N is the transmission matrix from the last interface $z = z_N$, respectively. Let $j < N$ and $\Delta z_j = z_{j+1} - z_j$, $j = 1, 2, \ldots, N-1$, is the layer thickness. Then by jumping across the layer boundary and using (12), (13) we obtain

$$U_j^- = J_{A,j}^T e^{i\omega\Lambda_j \Delta z_j} U_{j+1}^- - J_{B,j}^T e^{-i\omega\Lambda_j \Delta z_j} D_{j+1}^-$$
$$D_j^- = -J_{B,j}^T e^{i\omega\Lambda_j \Delta z_j} U_{j+1}^- + J_{A,j}^T e^{-i\omega\Lambda_j \Delta z_j} D_{j+1}^- \qquad (16)$$

Define reflection and transmission matrices Γ_j, T_j by

$$U_j^- = \Gamma_j D_j^-, U_j^+ = T_j D_j^- \qquad (17)$$

From (16), (17) we obtain by induction

$$\Gamma_j = \left(J_{A,j}^T \tilde{\Gamma}_{j+1} - J_{B,j}^T \right) \left(-J_{B,j}^T \tilde{\Gamma}_{j+1} + J_{A,j}^T \right)^{-1}$$
$$T_j = T_{j+1} e^{i\omega\Lambda_j \Delta z_j} \left(-J_{B,j}^T \tilde{\Gamma}_{j+1} + J_{A,j}^T \right)^{-1} \qquad (18)$$

where $\tilde{\Gamma}_{j+1} = e^{i\omega\Lambda_j \Delta z_j} \Gamma_{j+1} e^{i\omega\Lambda_j \Delta z_j}$, and Γ_{j+1} is symmetric. Thus, all the reflection and transmission matrices can be calculated by (18), starting with (15).

5. Sources and boundary conditions. Consider a $2n$-dimensional system of the form (5) with $^{(m)}$ omitted. Let the source be of the form

$$S = S_0 \delta(z - z_s) + S_1 \delta'(z - z_s) \qquad (19)$$

with S_0, S_1 independent of z.

Define n-vectors $S_A, S_B : \left(S_A, S_B\right)^T = i\omega M S_1 - S_0$. Using this formula we obtain the following jump condition across the source

$$\Phi(z_s^-) = \Phi(z_s^+) + \left(S_A, S_B\right)^T \qquad (20)$$

Inserting a fictitious layer boundary at $z = z_s^+$ we compute the reflection matrix $\Gamma_s \equiv \Gamma(z_s^+)$. Since the material properties do not change at z_s, we have

$$\Psi(z_s^+) = \left(\Gamma_s D_s, D_s\right)^T \qquad (21)$$

where $D_s \equiv D(z_s^+), U_s \equiv U(z_s^+)$. Using (11), (20) and (21) we obtain

$$\Psi(z_s^-) = \left(\Gamma_s D_s, D_s\right)^T + \frac{1}{\sqrt{2}}\left(L_2^T S_A + L_1^T S_B, L_2^T S_A - L_1^T S_B\right)^T$$

This expression may now be propagated upwards through layers, using (12) and jumped upwards across layers boundaries until we reach the free surface at $z = 0^+$. Then n boundary conditions at $z = 0$ can be used to find the n unknowns D_s.

Consider one particular case when $z_s \in (0, z_1)$. In this case

$$\Psi(0^+) = \left(e^{i\omega\Lambda z_s}\Gamma_s D_s, e^{-i\omega\Lambda z_s} D_s\right)^T +$$
$$+ \frac{1}{\sqrt{2}}\left(e^{i\omega\Lambda z_s}\left(L_2^T S_A + L_1^T S_B\right), e^{-i\omega\Lambda z_s}\left(L_2^T S_A - L_1^T S_B\right)\right)^T$$
$$(22)$$

Define

$$\Phi(0^+) = \left(G_A \Phi_0, G_B \Phi_0\right)^T \qquad (23)$$

For System 1, let

$$\Phi_0^{(1)} = \left(\tilde{v}_3, -\tilde{q}_3, \tilde{v}_1\right)^T_{z=0^+}$$

$$G_A^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, G_B^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We can check that (23) holds for System 1 with the boundary conditions $\tilde{\tau}_{13} = \tilde{\tau}_{33} = \tilde{p} = 0$ at the free surface $z = 0$.

For System 2, let

$$\Phi_0^{(2)} = \tilde{v}_2(0^+), \quad G_A^{(2)} = 1, G_B^{(2)} = 0$$

Then it may be checked that (23) holds for System 2 with the boundary condition $\tilde{\tau}_{23} = 0$ at the free surface $z = 0$. Now using (11), (22) and (23) we obtain

$$\Phi_0 = \left(e^{i\omega\Lambda z_s}\Gamma_s e^{i\omega\Lambda z_s}\left(L_2^T G_A - L_1^T G_B\right) - \left(L_2^T G_A + L_1^T G_B\right)\right)^{-1} \times$$
$$\times e^{i\omega\Lambda z_s}\left(\Gamma_s\left(L_2^T S_A - L_1^T S_B\right) - \left(L_2^T S_A + L_1^T S_B\right)\right)$$
$$D_s = \frac{1}{\sqrt{2}} e^{i\omega\Lambda z_s}\left(L_2^T G_A - L_1^T G_B\right)\Phi_0 - \frac{1}{\sqrt{2}}\left(L_2^T S_A - L_1^T S_B\right)$$
$$(24)$$

In particular, when $z_s = 0^+$ we get

$$\Phi_0 = \left((\Gamma_s - I)L_2^T G_A - (\Gamma_s + I)L_1^T G_B\right)^{-1} \times$$
$$\times \left((\Gamma_s - I)L_2^T S_A - (\Gamma_s + I)L_1^T S_B\right)$$
$$(25)$$

Φ_0 defines all of Φ at the free surface, and $D_s, U_s = \Gamma_s D_s$ give all of Φ just below the source. Now we are able theoretically to compute Φ in any $z \in \mathbb{R}_+$ by propagating trough the layers using (12) and (13).

Remark. *Propagation of an upward-going wave in the downward direction will be unstable numerically using (12), because the complex exponentials grow rather than decay with distance. Then numerically, one has to obtain U from D using Γ_j, or the transmission matrix T_j.*

Inverting the rotation transform, we can calculate the hat (^) variables, i.e.,

$$\hat{v} = \Omega^T \tilde{v}, \hat{q} = \Omega^T \tilde{q}, \hat{\tau} = \Omega^T \tilde{\tau}\Omega, \hat{p} = \tilde{p} \qquad (26)$$

The matrices for Systems 1 and 2 depend only on the magnitude k. However, the transformation (26) depends on k_1, k_2. For any function $\hat{\xi}(k)$ let

$$\Xi_{j_1,j_2}\left(\hat{\xi}\right) \equiv F_{x_1 x_2}^{-1}\left(k_1^{j_1} k_2^{j_2}\hat{\xi}(k)\right) = (-i)^{j_1+j_2}\partial_{x_1}^{j_1}\partial_{x_2}^{j_2} F_{x_1 x_2}^{-1}\left(\hat{\xi}(k)\right)$$

We can compute these quantities as Hankel transforms in the cylindrical coordinates r, θ, z. Define

$$B_{j_1,j_2}(\hat{\xi}) = \frac{1}{2\pi}\int_0^\infty k^{j_1} J_{j_2}(kr)\hat{\xi}(k) dk$$

where J_n is the Bessel function and j_1, j_2 are nonnegative integers. Then

$$\Xi_{0,0} = B_{1,0}, \Xi_{1,0} = i\cos\theta B_{2,1}, \Xi_{0,1} = i\sin\theta B_{2,1}$$

$$\Xi_{1,1} = \sin\theta\cos\theta\left(B_{3,0} - \frac{2}{r}B_{2,1}\right)$$

$$\Xi_{2,0} = \cos^2\theta B_{3,0} - \frac{\cos 2\theta}{r}B_{2,1}$$

$$\Xi_{0,2} = \sin^2\theta B_{3,0} + \frac{\cos 2\theta}{r}B_{2,1}$$
$$(27)$$

see White and Zhou (2006) for details. These formulas are used to get the solution in real space.

Examples

1. Dynamite source. A dynamite source imposed on the solid and the fluid can be defined in the following form

$$f(x) = g(x) = -h(\omega)\nabla\delta(x - x_s)$$

where δ is the Dirac function, $x_s = (0,0,z_s)$ is the source position and $h(\omega)$ is the spectrum of the seismic moment. Applying the Fourier transform $F_{x_1 x_2}$ we obtain

$$\hat{f} = \hat{g} = -h(\omega)\big(ik_1\delta(z - z_s), ik_2\delta(z - z_s), \delta'(z - z_s)\big)^T$$

and rotation by Ω yields

$$\tilde{f} = \tilde{g} = -h(\omega)\big(ik\delta(z - z_s), 0, \delta'(z - z_s)\big)^T \qquad (28)$$

Substitution of (28) into $S^{(1)}$ of (7) yields the source for System 1, in the form of (19), with

$$S_0^{(1)} = h(\omega)\left(0, ik - \frac{\omega\rho_f k\kappa}{\eta - i\omega\rho_E\kappa}, \frac{k^2\kappa}{\eta - i\omega\rho_E\kappa}, 0, 0, 0\right)^T$$

$$S_1^{(1)} = h(\omega)\big(0,0,0,1,0,-1\big)^T$$

$$(29)$$

Substitution of (28) into (7) shows that $S^{(2)}$ is zero, then $\dot{\tilde{u}}_2, \tilde{\tau}_{23}$ associated with System 2 are zero too. This is to be expected result because System 2 is related to SH-waves, which are not excited by the dynamite source. Substitution of (29) into $\big(S_A, S_B\big)^T = i\omega M S_1 - S_0$ gives

$$S_A^{(1)} = i\beta h(\omega)\big(\omega(C - M), 2kG(M - C), \omega(\lambda + 2G - C)\big)^T$$

$$S_B^{(1)} = \big(0,0,0\big)^T$$

$$(30)$$

Formulas (30) may be used in (24) or (25) for a shallow source, to obtain all the tilde (~) functions. To invert rotation Ω, using (26), note that from (8) and the vanishing of System 2, $\tilde{v}_2, \tilde{q}_2, \tilde{\tau}_{12}, \tilde{\tau}_{23}$ are identically zero.

All the remaining tilde (~) functions depend of k only and can be calculated by the following formulas

$$\hat{v}_1 = \frac{k_1}{k}\tilde{v}_1, \hat{v}_2 = \frac{k_2}{k}\tilde{v}_1, \hat{v}_3 = \tilde{v}_3$$

$$\hat{q}_1 = \frac{k_1}{k}\tilde{q}_1, \hat{q}_2 = \frac{k_2}{k}\tilde{q}_1, \hat{q}_3 = \tilde{q}_3$$

$$\hat{\tau}_{11} = \frac{k_1^2\tilde{\tau}_{11} + k_2^2\tilde{\tau}_{22}}{k^2}, \hat{\tau}_{12} = \frac{k_1 k_2(\tilde{\tau}_{11} - \tilde{\tau}_{22})}{k^2} \qquad (31)$$

$$\hat{\tau}_{22} = \frac{k_2^2\tilde{\tau}_{11} + k_1^2\tilde{\tau}_{22}}{k^2}, \hat{\tau}_{13} = \frac{k_1\tilde{\tau}_{13}}{k}$$

$$\hat{\tau}_{23} = \frac{k_2\tilde{\tau}_{13}}{k}, \hat{\tau}_{33} = \tilde{\tau}_{33}, \hat{p} = \tilde{p}$$

Then the Fourier transform $F_{x_1 x_2}$ can be inverted in cylindrical coordinates (r, θ, z) using (27) to obtain the solid and fluid velocities

$$v = \big(iB_{1,1}(\tilde{v}_1)\big)e_r + \big(B_{1,0}(\tilde{v}_3)\big)e_z$$

$$q = \big(iB_{1,1}(\tilde{q}_1)\big)e_r + \big(B_{1,0}(\tilde{q}_3)\big)e_z$$

$$(32)$$

where e_r, e_z are unit vectors in the r, z coordinate directions, respectively, and the stress tensor components and the pressure

$$\tau_{11} = \Xi_{2,0}(k^{-2}\tilde{\tau}_{11}) + \Xi_{0,2}(k^{-2}\tilde{\tau}_{22}), \tau_{12} = \Xi_{1,1}(k^{-2}(\tilde{\tau}_{11} - \tilde{\tau}_{22}))$$

$$\tau_{22} = \Xi_{0,2}(k^{-2}\tilde{\tau}_{11}) + \Xi_{2,0}(k^{-2}\tilde{\tau}_{22}), \tau_{13} = \Xi_{1,0}(k^{-1}\tilde{\tau}_{13})$$

$$\tau_{23} = \Xi_{0,1}(k^{-1}\tilde{\tau}_{13}), \tau_{33} = \Xi_{0,0}(\tilde{\tau}_{33}), p = \Xi_{0,0}(\tilde{p})$$

$$(33)$$

2. Vertical source. We next consider a vertical point source acting on the free surface $z = 0$, i.e.,

$$f(x) = g(x) = (0,0,1)^T h(\omega)\delta(x_1)\delta(x_2)\delta(z - z_s)$$

where $z_s \to 0^+$ puts the force on the free surface. This models hammer, weight drop, and vibroseis sources. Applying the Fourier transform $F_{x_1 x_2}$ and rotation Ω we arrive at

$$\tilde{f} = \tilde{g} = \hat{f} = \hat{g} = (0,0,1)^T h(\omega)\delta(z - z_s) \qquad (34)$$

Substitution of (34) into (7) yields the source for Systems 1 and 2 in the form

$$S^{(1)} = (0,0,0,-1,0,1)^T h(\omega)\delta(z - z_s), S^{(2)} = (0,0)^T$$

Thus, all the variables in System 2 are zero, as it was in the case of dynamite source. From (19) and definition of S_A, S_B we obtain

$$S_A^{(1)} = \big(0,0,0\big)^T, S_B^{(1)} = \big(1,0,-1\big)^T h(\omega)$$

Now all the tilde variables at the free surface may be computed using (25) as $z_s \to 0^+$ and propagated anywhere else in space. Note that $S_A^{(1)}, S_B^{(1)}$ are independent of k_1, k_2, so the tilde variables depend only on k and not on wave number direction. Therefore, similar to dynamite we can transform to the hat variables using (31) and transform back to the spatial variables using (32) and (33).

Conclusion
Based on the Ursin method, we have derived explicit formulas of the solution to a boundary-value problem formulated for Biot's system, which can be used as the basis of a numerical algorithm and study of the propagation of elastic waves in porous plane-layered media.

Acknowledgments

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Appendix

System 1. There are three modes: fast compressional wave ($\lambda_1^{(1)}$), Biot slow wave ($\lambda_2^{(1)}$), and vertical shear wave ($\lambda_3^{(1)}$).

Eigenvalues:

$$(\lambda_j^{(1)})^2 = -\gamma^2 + \beta\left(C\rho_f - \frac{M\rho}{2} - i(\lambda+2G)\frac{\eta - i\omega\rho_E\kappa}{2\omega\kappa}\right) \pm \frac{\beta}{2}\sqrt{\left(i(\lambda+2G)\frac{\eta-i\omega\rho_E\kappa}{\omega\kappa} - M\rho\right)^2 - 4\left(M\rho_f - iC\frac{\eta-i\omega\rho_E\kappa}{\omega\kappa}\right)\left(C\rho - (\lambda+2G)\rho_f\right)}, j=1,2$$

(with (+) for m=1 and (-) for m=2), and $(\lambda_3^{(1)})^2 = -\gamma^2 + G^{-1}\left(\rho + \frac{i\omega\kappa\rho_f^2}{\eta - i\omega\rho_E\kappa}\right)$

Eigenvectors:

$$a_j^{(1)} = \bar{a}_j\left(-1, 2G\gamma, \zeta_j\right)^T, j=1,2, a_3^{(1)} = \frac{\bar{a}_3}{\lambda_3^{(1)}}\left(\gamma, G(\lambda_3^{(1)})^2 - G\gamma^2, -\frac{i\omega\kappa\gamma\rho_f}{\eta - i\omega\rho_E\kappa}\right)^T$$

$$b_j^{(1)} = \frac{\bar{a}_j}{\lambda_j^{(1)}}\left(2G\gamma^2 - \rho - \rho_f\zeta_j, \gamma, \rho_f + i\zeta_j\frac{\eta - i\omega\rho_E\kappa}{\omega\kappa}\right)^T, j=1,2, b_3^{(1)} = \bar{a}_3\left(2G\gamma, 1, 0\right)^T$$

where

$$\zeta_j = \frac{C\rho - (\lambda+2G)\rho_f}{\frac{(\lambda_j^{(1)})^2 + \gamma^2}{\beta} - C\rho_f + i(\lambda+2G)\frac{\eta - i\omega\rho_E\kappa}{\omega\kappa}}, \bar{a}_j = \sqrt{\frac{\lambda_j^{(1)}}{\rho + 2\rho_f\zeta_j + i\zeta_j^2\frac{\eta - i\omega\rho_E\kappa}{\omega\kappa}}}, j=1,2, \bar{a}_3 = \sqrt{\frac{\lambda_3^{(1)}}{G\gamma^2 + G(\lambda_3^{(1)})^2}}$$

System 2. There is the horizontal shear wave mode only.

$$(\lambda^{(2)})^2 = -\gamma^2 + G^{-1}\left(\rho + \frac{i\omega\kappa\rho_f^2}{\eta - i\omega\rho_E\kappa}\right), a^{(2)} = \sqrt{\frac{1}{G\lambda^{(2)}}}, b^{(2)} = \sqrt{G\lambda^{(2)}}$$