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HOW TO OVERCOME NUMERICAL INSTABILITY WHEN THE DENSITY JUMP BETWEEN LAYERS IS CONSTANT

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It has been observed in oceanic modelling that numerical instability is caused when the density jump between layers is constant. This instability is usually overcome by using variable density jumps, according to McNider & O'Brien (1973). In decomposing the vertical modes of the linearized model equation, it is shown that the eigenvectors of the layered system may introduce an anomalous fast mode if the parameters are not adequately chosen.

Em modelagem oceânica se observou que a instabilidade numérica é causada quando o salto de densidade entre camadas é constante. Este tipo de instabilidade pode ser superada ao utilizar saltos de densidade variáveis de acordo com McNider & O'Brien (1973). Ao decompor os modos verticais das equações linearizadas se demonstra que os autovalores do sistema podem introduzir um modo anômalo se os parâmetros não são escolhidos adequadamente.

INTRODUCTION

During the course of previous research (Camerlengo, 1982), it was verified that if the jump in density between layers remains constant at a certain time interval, the numerical model becomes numerically unstable. The problem is traditionally overcome by using variable density jumps between layers (McNider & O'Brien, 1973). However, a deeper understanding of the occurrence of such numerical instability is required. To achieve this goal, the phase speed of the first baroclinic (the fastest) mode is evaluated.

EQUATIONS OF MOTION

To accomplish this thrust, a linearized version of the nonviscous model equation is sufficient. Following Camerlengo (1982), a linearized version of the momentum equations, after filtering the barotropic mode, has the following approximate expression:

$$\frac{\partial u^j}{\partial t} - f v^j = -g \sum_{i=1}^{4} \left(\frac{\rho^5 - \rho^\nu}{\rho^j}\right) \frac{\partial h_i}{\partial x} \tag{1}$$

$$\frac{\partial v^j}{\partial t} + f u^j = -g \sum_{i=1}^{4} \left(\frac{\rho^5 - \rho^\nu}{\rho^j}\right) \frac{\partial h_i}{\partial y} \tag{2}$$

where the superscript j goes from 1 at the top to 5 at the bottom layer and ν is the maximum value of the pair (i,j). The linear form of the continuity equation leads to:

$$\frac{\partial h^k}{\partial t} + H^k \left[\frac{\partial u^k}{\partial x} + \frac{\partial u^k}{\partial y}\right] = 0 \tag{3}$$

where H^k represents the mean depth of the k.th layers. The differentiation of (1) with respect to time yields:

$$\frac{\partial^2 u^j}{\partial t^2} - f \frac{\partial v^j}{\partial t} = -g \sum_{1=i}^{4} \left[\left(\frac{\rho^5 - \rho^\nu}{\rho^j}\right) \frac{\partial}{\partial x}\left(\frac{\partial h_i}{\partial t}\right)\right] \tag{4}$$

Introducing (3) into this last expression yields:

$$\frac{\partial^2 u^j}{\partial t^2} - f \frac{\partial v^j}{\partial t} = -g \sum_{1=i}^{4} \left[\left(\frac{\rho^5 - \rho^\nu}{\rho^j}\right) H^k \left(\frac{\partial^2 u^k}{\partial x^2} + \frac{\partial^2 u^k}{\partial x \partial y}\right)\right] \tag{5}$$

This set of equations is mathematically coupled and therefore represents a physical interdependence.

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To decouple (5), u^j and v^j can be decomposed into its vertical modes in the following manner:

$$(\;)^j = \sum_{k=1}^{5} c^j_k \, (-)_k \qquad (6)$$

where the value $k = 5$ corresponds to the barotropic mode while $k = 1, 2, 3$, and 4 represents the four possible baroclinic modes. On the other hand, c^j_k and u_k represent the eigenvector and the amplitude of the k-th mode, respectively.

The elements of a certain matrix N_{kj} may be defined in the form:

$$N_{kj} = gH_k \left(\frac{\rho^5 - \rho^\nu}{\rho^j}\right) \qquad (7)$$

Upon substitution of (6) and (7) in (5), a single expression for each mode is obtained:

$$c^j_k \left(\frac{\partial^2 \bar{u}_k}{\partial t^2} - f \frac{\partial \bar{v}_k}{\partial t}\right) = \sum_{k=1}^{5} N_{kj} \, c^j_k \left(\frac{\partial^2 \bar{u}_k}{\partial x^2} + \frac{\partial^2 \bar{u}_k}{\partial x \partial y}\right) \qquad (8)$$

Because c^j_k is an eigenvector of the matrix N_{kj}, it must satisfy the relation:

$$\sum_{k=1}^{5} N_{kj} \, c^j_k = \alpha_k \, c^j_k \qquad (9)$$

where α_k is the associated eigenvalue. Using (9) in (8) the final expression for the linear u momentum equation is

$$\frac{\partial^2 \bar{u}_k}{\partial t^2} - f \frac{\partial \bar{v}_k}{\partial t} = \alpha_k \left(\frac{\partial^2 \bar{u}_k}{\partial x^2} + \frac{\partial^2 \bar{u}_k}{\partial x \partial y}\right). \qquad (10)$$

Proceeding in the same manner, a similar expression may be derived for the v momentum equation:

$$\frac{\partial^2 \bar{v}_k}{\partial t^2} + f \frac{\partial \bar{u}_k}{\partial t} = \alpha_k \left(\frac{\partial^2 \bar{v}_k}{\partial x \partial y} + \frac{\partial^2 \bar{v}_k}{\partial y^2}\right) \qquad (11)$$

Equations (10) and (11) are no longer coupled. From (7) it follows that the elements N_{5j} and N_{k5} are identically zero. This result should be no surprise as the barotropic mode has been filtered out (Camerlengo, 1982). It is convenient to define a matrix M_{kj} which is related to the matrix N_{kj} by the form:

$$M_{kj} = N_{kj} \qquad (12)$$

where k and j vary from one to four in M_{kj}.

The next task is to evaluate the eigenvalues of the first baroclinic mode. Two cases are considered. In one case the densities are arbitrarily chosen to be $\rho_1 = 1.0000$, $\rho_2 = 1.0020$, $\rho_3 = 1.0022$, $\rho_4 = 1.0025$ and $\rho_5 = 1.0027$ g cm^{-3}. In the other case, the density jump between layer is constant from layer to layer. Thus, $\rho_1 = 1.000$ g cm^{-3}, $\rho_{1+i} = \rho_1 + i\Delta\rho$, where $i = 1, ..., 4$ and $\Delta\rho = 10^{-3}$ g cm^{-3}. In both cases, the layer depths are arbitrarily chosen to be:

$$H_1 = mk, \quad H_2 = nk, \quad H_3 = H_4 = k, \qquad (13)$$

where k = 50 meters and m and n are allowed to vary alternatively from one to three. In all the different test conducted, the eigenvalues of the first baroclinic mode proved to be larger than in the case of equal jump in density; that difference ranged between 80% and 140% (Table 1).

Table 1. Eigenvalues of the first baroclinic mode for $H_1 = mk$, and $H_2 = nk$, where m and n vary from 1 to 3 and, when: a) $\rho_1 = 1.0$; $\rho_2 = 1.002$; $\rho_3 = 1.0022$; $\rho_4 = 1.0025$; $\rho_5 = 1.0027$ g cm^{-3}; b) $\rho_{1+i} = \rho_1 + i\Delta\rho_i$, where $i = 1, ..., 4$ and $\Delta\rho_i = 10^{-3}$ g cm^{-3}.

	H_2		
H_1	1	2	3
1	14388.9	14859.0	15886.9
2	27678.5	27800.8	27973.7
3	41124.3	41192.5	41273.7

(a)

	H_2		
H_1	1	2	3
1	34534.3	44855.7	57612.9
2	50452.5	56025.1	64769.9
3	68917.0	72054.1	77099.3

(b)

SUMMARY AND CONCLUSIONS

The phase speed is equal to the square root of the eigenvalue of the respective baroclinic mode. Therefore, the phase speed of the first baroclinic mode is larger for the case where jump in density is constant. It is immediately concluded, from the Courant-Friedrichs-Levy (CFL) criterion for numerical instability, that a smaller time step, Δt, for this case is required. This result was confirmed in all the model runs tested. In particular, the time step used in the multi-layered numerical model (Camerlengo, 1982) is 20% larger than the one which would be needed if an equal density jump between layers were used.

Thus, an old problem of instability in a multi-layered system has been solved. Namely, that the eigenvectors of a layered system may introduce an anomalous fast mode if the parameters are not properly chosen.

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