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ATMOSPHERIC REFRACTION FROM THE ZENITH TO THE HORIZON

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In this paper the role of the atmospheric refraction in the measurements of positions of artificial satellites and celestial objects is investigated. An analytically approximated expression for the calculation of the atmospheric refraction angle as a function of the zenithal angle referred to the position of the satellite or the celestial object is presented. The formula obtained is valid for any virtual position (from the zenith to the astronomical horizon). The theoretical predictions are then compared with values obtained with other methods and published elsewhere. It is found that there is a maximum error of 6 per cent between these theoretical predictions and the available published data. The analytical results obtained by direct integration are in good agreement with the results obtained by numerical computation, using standard numerical methods. The formula obtained is particularly appropriate for the determination of the refraction angle of electromagnetic waves in the neighbourhood of the astronomical horizon. The influence of temperature gradients is also investigated. A discussion is presented on the maximum fluctuation of the atmospheric refraction when the direction of propagation of the electromagnetic wave is orthogonal to the direction of the temperature gradient.

REFRAÇÃO ATMOSFÉRICA DO ZÊNITE ATÉ O HORIZONTE – Investigamos o efeito da refração atmosférica na determinação da posição aparente de um astro ou de um satélite artificial da Terra. Determinamos analiticamente uma expressão aproximada para o cálculo da refração atmosférica em função do ângulo zenital referente à posição aparente do astro ou satélite. A fórmula deduzida neste trabalho é válida para qualquer valor do ângulo zenital (desde zero até 90^o, no horizonte). Os valores obtidos mediante aplicação desta fórmula concordam razoavelmente com resultados semelhantes que já existem na literatura (obtidos por outros métodos). Verificamos que os resultados analíticos baseados no método desenvolvido neste trabalho concordam satisfatoriamente com os resultados decorrentes dos cálculos feitos no computador mediante aplicação de métodos usuais para a integração numérica. Mostramos que a fórmula aproximada obtida é particularmente adequada para a determinação da refração decorrente de ondas eletromagnéticas nas vizinhanças do horizonte astronômico. Discutimos a influência da temperatura sobre as variações da refração atmosférica em função do ângulo zenital. Estimamos alguns valores para as flutuações máximas da refração atmosférica quando a direção da incidência da onda eletromagnética é ortogonal à direção do gradiente de temperatura.

1. INTRODUCTION

The measurements of artificial satellites and celestial objects positions are limited by two important corrections. The first one deals with the relative motion between the Earth and the celestial object considered. The second is related with the so called *atmospheric refraction* or *astronomic refraction* which is produced by the refraction of light in the atmosphere. In the present work we will develop an alternative theoretical approach in order to study the refraction of an electromagnetic wave in the Earth's atmosphere.

Let us consider the observer's eye situated at the point O as indicated in Fig. 1. The light ray which comes from the celestial object A follows the trajectory AO. The bending of the curve AO is downward, because the refractive index grows continuously along the path AO. Then, the celestial object appears to be at a higher altitude A' than the real position A.

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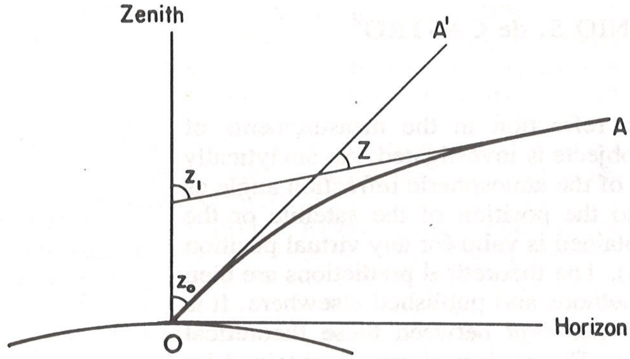

Figure 1

The difference between the virtual position and the real position is given by

$$Z = z_1 - z_0$$

where z_0 is the zenithal angle associated with the virtual position A' and z_1 is the angle associated with the real position A. The correction Z is named *astronomic refraction angle*, since it is to be applied to astronomic observations involving objects outside the atmosphere. If the objects were inside the effective atmosphere, the correction Z would be called *atmospheric refraction angle*. Henceforward, we will use the expression astronomic refraction angle to denote the above-mentioned correction Z.

Ptolemy was the first investigator to develop a theory about the astronomic refraction. Cassini, supposing a constant atmospheric density, found an astronomic refraction angle of 20 minutes at the astronomical horizon ($z_0 = 90^o$). Newton, supposing an isothermal atmosphere, concluded that the maximum astronomic refraction angle is equal to 40 minutes (at the horizon).

Modern approaches to the astronomic refraction problem are given by Danjon (1952), Garfinkel (1967), Mueller (1977) and Smart (1977).

According to Danjon (1952), the astronomic refraction angle can be calculated by the expression

$$Z = 60.34 \frac{273P}{(t + 273)\, 76} \tan z_0 \qquad (1)$$

where Z is given in seconds, P is the pressure in cm Hg and t is the temperature in ^oC. Under standard atmospheric conditions (P = 76 cm Hg; t = 0 ^oC) eq. (1) becomes

$$Z = 60.34 \tan z_0 \qquad (2)$$

We can verify that the eqs. (1) and (2) are not valid in the neighbourhood of the astronomical horizon. Equations (1) and (2) do not agree with the experimental results for angles z_0 greater than 70^o. The main objective of this work is to investigate the astronomic refraction in the neighbourhood of the astronomical horizon. In Sections 2 and 3 we will consider an isothermal atmosphere. In Section 4 we will investigate the influence of the temperature variations on the determination of the astronomic refraction angle.

2. CALCULATION OF THE ASTRONOMIC REFRACTION ANGLE

Fermat's principle is appropriate to the study of the propagation of a light ray in a material medium. According to this fundamental principle we obtain the following differential equation for the trajectory of a light ray in a transparent medium:

$$\frac{1}{r_0} = \frac{1}{n} \hat{N} \,.\, \text{grad } n \qquad (3)$$

where r_0 is the curvature radius of the trajectory, n is the refractive index of the medium and $\hat{N}$ is the unit vector orthogonal to the trajectory at the point where the curvature radius is r_0. The demonstration of the eq. (3) is given by Sommerfeld (1954).

By considering the symmetry of the Earth's gravitational field, we can suppose the atmosphere to be spherically symmetrical. Therefore, the refractive index n is a function which depends only on the distance r between the considered point and the Earth's centre. By use of the eq. (3) Born & Wolf (1959) obtained the following relation for the trajectory of the light ray:

$$nr \sin \phi = \alpha = \text{constant} \qquad (4)$$

where n is a function of r, α is a constant at every point of the trajectory of the light ray, ϕ is the angle between the tangent to the trajectory at a certain point A and the radius vector which links this point with the Earth's centre. Equation (4) is the implicit form of the equation of the trajectory of the light ray.

Figure 2 shows the trajectory AO of a light ray which arrives at the observer's eye O.

From Fig. 2 it is evident that

$$Z = \theta + \phi - z_0$$

Differentiating the previous equation with respect to r yields

$$\frac{dZ}{dr} = \frac{d\theta}{dr} + \frac{d\phi}{dr} \qquad (5)$$

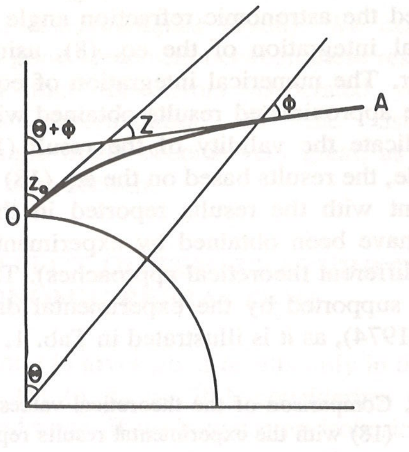

Figure 2

Equation (4) can be written as

$$nr \sin \phi = n_1 R_o \sin z_o \tag{6}$$

where R_o is the Earth's radius, n_1 is the refractive index at the Earth's surface and z_o is the zenithal angle of the virtual position of the celestial object. By using polar coordinates we get

$$r \frac{d\theta}{dr} = \tan \phi \tag{7}$$

From the eqs. (5), (6) and (7), we obtain

$$dZ = - \frac{\alpha \, dn}{n(n^2r^2 - \alpha^2)^{1/2}} \tag{8}$$

By using eq. (8), Danjon (1952) obtained the following result (originally derived by Laplace):

$$Z = 60.27 \tan z_o - 0.0669 \tan^3 z_o \tag{9}$$

where the astronomic refraction angle ought to be computed in seconds. Equation (9) gives the astronomic refraction angle Z as a function of the zenithal angle z_o, in the case of the standard atmospheric conditions. We must stress that Z diverges to infinity at the astronomical horizon.

Danjon (1952) has obtained the following expression for the determination of the astronomic refraction angle in the neighbourhood of the horizon.

$$Z = \alpha_o [1 - \alpha_o/2] \sqrt{2/a} \sin z_o \exp [X^2] \int_X^\infty dx \exp(-x^2)$$

where $X = (\cos z_o)/\sqrt{2a}$; $\alpha_o = 60''.343$ and $a = 0.0011078$.

In this Section we will develop and alternative approach for the determination of the astronomic refraction angle from the zenith to the horizon. By inserting $u = nr$ into eq. (8), we find that

$$dZ = - \frac{\alpha \, du}{u(u^2 - \alpha^2)^{1/2}} + \frac{\alpha \, dr}{r(n^2r^2 - \alpha^2)^{1/2}} \tag{10}$$

In order to obtain Z it is necessary to integrate eq. (10) from the Earth's surface, where $r = R_o$ and $n = n_1$, to a certain height H, where the refractive index n is equal to n_2. The integration of the first term of the second member of eq. (10) yields

$$\int_{u_1}^{u_2} \frac{\alpha \, du}{u(u^2 - \alpha^2)^{1/2}} = \cos^{-1} \left[\frac{\alpha}{n_2(R_o + H)}\right] - \cos^{-1} \left[\frac{\alpha}{n_1 R_o}\right] \tag{11}$$

where $u_1 = n_1 R_o$ and $u_2 = n_2(R_o + H)$.

In order to integrate the second term of the second member of eq. (10) it is necessary to know the exact relation between n and r. For the sake of simplicity we shall suppose an isothermal atmosphere (temperature gradients are considered in Section 4) and taking into account Boltzmann's distribution, we are able to write

$$\rho = \rho_o \exp(- \rho_o gh/P_o)$$

where ρ is the atmospheric density at a height h, ρ_o is the atmospheric density at the ocean level, g is the local value of the acceleration of the gravity and P_o is the standard value of the atmospheric pressure. By Gladstone and Dale's law, we can write

$$n = 1 + k\rho \tag{12}$$

where k is a constant. By inserting the expression for the atmospheric density ρ into eq. (12) we obtain

$$n = 1 + n_o \exp(- Bh) \tag{13}$$

where $B = \rho_o g/P_o$. On the Earth's surface and under standard atmospheric conditions, we have

$$n_1 = 1.0002925; \ \rho_o = 1.2932 \text{ kg/m}^3;$$

$$P_o = 1.013 \times 10^5 \text{ N/m}^2 \tag{14}$$

where $n_1 = 1 + n_o$. By using the value $g = 9.81$ m/s^2 we get

$$B = 1.2523 \times 10^{-4} \ m^{-1} \qquad (15)$$

In order to integrate the second term of the second member of the eq. (10) it is convenient to make a development in a Taylor series. After some algebraic transformations, we obtain

$$I = \int_{R_o}^{R_o+H} \frac{\alpha \, dr}{r(n^2r^2 - \alpha^2)^{1/2}} =$$

$$\int_o^H \frac{\alpha \, dh}{(R_o + h)(C + Dh)^{1/2}}$$

where

$$C = n_1 R_o^2 \cos^2 z_o \ ;$$

$$D = 2R_o n_1^2 - 2n_o n_1 B R_o^2 \qquad (16)$$

The previous integral yields the result

$$I = \frac{2n_1 R_o \sin z_o}{(R_o D - C)^{1/2}} [\tan^{-1} [\frac{C + DH}{R_o D - C}] -$$

$$\tan^{-1} [\frac{C}{R_o D - C}]] \qquad (17)$$

From eqs. (10), (11) and (17) we find the desired relation between the astronomic refraction angle and z_o:

$$Z = I + \cos^{-1} (\sin z_o) - \cos^{-1} (E/F) \qquad (18)$$

where

$$E = n_1 R_o \sin z_o \ ; \ F = n_2 (R_o + H)$$

In this approximation, H is much smaller than the Earth's radius. On the other hand, from eq. (13), we conclude that the refractive index decays exponentially with h. In dealing with exponential decreasing, it is a standard procedure to take an effective value which becomes 1/e of the maximum value of the function. We will choose a value of H such that the term n_o exp (-Bh) becomes equal to n_o/e. Thus,

$$H = 1/B = 7985 \ m \qquad (19)$$

By using the mean Earth's radius $R_o = 6371$ km and eqs. (14), (15), (16), (17), (18) and (19) we have calculated the function $Z = Z(z_o)$. We have also calculated the astronomic refraction angle Z by direct numerical integration of the eq. (8), using a digital computer. The numerical integration of eq. (8) yields the same approximated results obtained with eq. (18). This indicate the validity of the result (18). On the other side, the results based on the eq. (18) are in good agreement with the results reported in the literature (which have been obtained by experimental methods and by different theoretical approaches). These results are also supported by the experimental data reported by Pan (1974), as it is illustrated in Tab. 1.

Table 1. Comparison of the theoretical values based in eq. (18) with the experimental results reported by Pan (1974).

z_o	eq. (18)	Experiment
60°	1'06"	1'40"
65°	1'22"	2'04"
70°	1'45"	2'37"
75°	2'22"	3'34"
80°	3'29"	5'16"
85°	7'05"	9'45"

3. ASTRONOMIC REFRACTION IN THE NEIGHBOURHOOD OF THE HORIZON

Danjon (1952) obtained for $z_o = 90°$ an astronomic refraction angle equal to 37' 52" (under standard conditions). For $z_o = 90°$, according to the eq. (18) we obtain Z = 36.7 minutes (under standard conditions).

The values of $Z = Z_1$ in the neighbourhood of the horizon, obtained from eq. (18) are indicated in Tab. 2. In this table the results Z_1 from $z_o = 89°$ to $z_o = 90°$ are compared with the results $Z = Z_2$ reported

Table 2. Comparison of the theoretical values based in eq. (18) with the results report by *Éphémérides Astronomiques* (1986).

z_o	Z_1	Z_2
89°00'	23'27"	25'37"
89°10'	25'17"	27'03"
89°20'	27'19"	28'38"
89°30'	29'34"	30'21"
89°40'	32'01"	32'14"
89°50'	34'43"	34'19"
90°00'	37'38"	36'36"

Éphémerides Astronomiques (1986). We can verify that there is good agreement between these results. We must stress that the errors associated with the experimental determination of the astronomic refraction angle might become very great, as we will show in the next Section.

4. THE EFFECT OF TEMPERATURE ON THE ASTRONOMIC REFRACTION

Equation (1) gives good results only in the range from $z_o = 0^o$ to $z_o = 70^o$, considering constant standard conditions. If we suppose an isobaric change Δt of the temperature, according to eq. (1), we get

$$\frac{\Delta Z}{Z} = - \frac{\Delta t}{t + 273}$$

Therefore, for a change $\Delta t = 1\ ^oC$, we obtain

$$\Delta Z/Z = - 1/273$$

The previous relation gives the variation of the astronomic refraction angle for a variation of $1\ ^oC$ in the neighbourhood of $t = 0\ ^oC$. Since eq. (1) is valid just in the range from $z_o = 0^o$ to $z_o = 70^o$, we conclude that the previous method should be only used in this range. Henceforth, we will study the effect of the temperature variations on the astronomic refraction angle in the neighbourhood of the horizon.

According to Fermat's principle, the trajectory of the light ray obeys eq. (3). By supposing a celestial object at the horizon, the direction AO indicated in Fig. 1 is orthogonal to the zenithal direction referring to the observer's position O. Thus, we can write expression (3) as

$$\frac{1}{r_o} = \frac{1}{n}\frac{dn}{dN} \qquad (20)$$

where dn/dN is the gradient of the refractive index (normal derivative of n in the direction orthogonal to the direction of the incident ray). We want to know what happens when temperature variations occur in the atmosphere in the neighbourhood of the observer's position. From the ideal gases law, we have

$$\rho = PM/RT \qquad (21)$$

where M is the mean molecular mass, T is the absolute temperature and R is the ideal gas constant. By differentiating eq. (21) with respect to N, we obtain

$$\frac{d\rho}{dN} = - \frac{PM}{RT^2}\frac{dT}{dN} \qquad (22)$$

Using eq. (12) we get

$$\frac{1}{n}\frac{dn}{dN} = \frac{k}{n}\frac{d\rho}{dN} \qquad (23)$$

From eqs. (20), (22) and (23), we have

$$\frac{1}{r_o} = - \frac{(n-1)\rho}{nT}\frac{dT}{dN} \qquad (24)$$

Substituting the values $n = 1.0002925$, $T = 273\ ^oK$ and $dT/dN = 1\ ^oC$ in eq. (24) we find $r_o = 10$ km. Thus, an isobaric temperature gradient of $1\ ^oC$ in the direction orthogonal to the horizontal direction at the observer's position, produces a great bending in the trajectory of the light ray. When the temperature gradient occurs in such a manner that the upper atmospheric layer is warmer than the inferior one, the bending of the trajectory of the light ray is downward. In the opposite case, the bending is upward. The last case occurs in deserts and gives rise to the effect of mirage.

In the circumstance described above we conclude that a temperature gradient of about $1\ ^oC$ produces an angular correction of 20 minutes in the computation of the astronomic refraction angle. The temperature corrections should be considered not only in observations of satellites and celestial objects but also in every precise observation which needs optical systems immersed in the atmospheric air.

5. CONCLUSIONS

Equations (1), (2) and (9) are useful only in the range from $z_o = 0^o$ to $z_o = 70^o$. In this work we obtained the result (18) which is valid for the entire range from $z_o = 0^o$ to $z_o = 90^o$. This result is particularly useful for the determination of the astronomic refraction angle in the neighbourhood of the horizon, as we can see in Tab. 2.

In the previous Sections we discussed the influence of the temperature variations in the determination of the astronomic refraction angle. If the observation of a satellite or a celestial body were made in the neighbourhood of the zenith, the temperature corrections would not be important at all. However, if the observation is made in the neighbourhood of the horizontal direction the temperature corrections will be very important. On the other hand, we conclude that the measurements of celestial bodies positions in the neighbourhood of the astronomical horizon are very difficult, since the temperature fluctuations which occur in the atmosphere produce a great correction in the astronomical refraction angle. The role of temperature and pressure variations in the determination of the atmospheric and astronomic refraction angle is a matter which needs further theoretical research.

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