doi: 10.22564/rbgf.v10i1.1119

EQUATIONS OF CONSERVATION LAWS IN THE INTERPLANETARY COLLISIONFREE PLASMA

P. Alexander e S. N. Duhau

Laboratorio de Aeronomía y Geomagnetismo,
Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires

A two fluid model for a collisionfree plasma (as is the case of the solar wind starting from about 0.1 AU) that includes two new energy equations for the electrons, which has been recently introduced by one of the authors, is considered to derive an equivalent set of equations, but now expressed in conservation form. These equations do not only refer as usual to mass, momentum and energy, but also to other combinations of variables. We show the relation between the different physical quantities and how the electrons and protons are coupled through the momentum equation. Finally, we apply these relations to study the constants of motion of the expanding solar fluid.

EQUAÇÕES PARA LEIS CONSERVATIVAS NO PLASMA INTERPLANETÁRIO SEM COLISÕES - Considera-se um modelo de dois fluidos para um plasma rarefeito (sem colisões, como é o caso para o vento solar que começa em torno de 0,1 UA), que inclue duas novas equações de energia para os elétrons, recentemente incluídas por um dos autores, e que são usadas para deduzir um conjunto equivalente de equações expressas numa forma conservativa. Estas equações além de se referir como de costume à massa, momento, e energia, também o fazem a outras combinações de variáveis. Mostra-se a relação entre diferentes quantidades físicas e como elétrons e prótons são acoplados entre si através da equação de momento. Finalmente aplicam-se estas relações para o estudo das constantes de movimento para o fluido solar em expansão.

1. INTRODUCTION

The Chew, Goldberger & Low (1956) equations (CGL) are a one-fluid system for the thermodynamic variables of the ions, that are coupled to the electrons only through the electromagnetic variables and they have been widely used to describe interplanetary hydromagnetic phenomena.

The magnitude of those variables in a collisionless plasma has been re-examined by one of the present authors (Duhau 1984) and a two-fluid equation system in the limits in which the Larmor radius to mean free path ratio $\varepsilon \rightarrow 0$ (MHD approximation) and the electron-to-ion mass ratio $\alpha \rightarrow 0$ has been found from the expansion of the Vlasov equation. Since the electric charge scales as ε^{-1} it should be noticed that both limits do not imply that the electron mass $\rightarrow 0$, which would lead to a one-fluid system (see Grad, 1967; Duhau, 1984). It has been shown that the first order electric field contributes to the equation of motion of ions (one of the assumptions underlying the CGL equations is that it can be assumed to be null) and electrons with a zero order term, providing a coupling mechanism between the thermodynamic variables of both species. To determine the electronic pressure it is necessary now to close the system of equations for these species. The energy equations of the electrons must be included in the equation set and to close this system a simple mathematical representation of the measured quasi-stationary velocity distribution function of this species (Feldman et al., 1975) is used. Finally, it is shown that the heat is mainly transported by the electrons and it is considered, as suggested by the satellite data, that the electrons' thermal anisotropy is small, so that to a first approximation the heat transported by the ions and the electrons' anisotropy may be neglected.

In the present paper the equations of the model are combined to derive an equivalent set of equations, but now expressed in conservation form. These new equations are applied to find the constants of motion of the solar coronal expansion.

2. THE TWO-FLUID MODEL EQUATION SET

$$n = zN \tag{1}$$

$$\frac{\partial N}{\partial t} + \nabla \cdot (N\boldsymbol{u}) = 0 \tag{2}$$

$$MN\frac{d\boldsymbol{u}}{dt} + \nabla \cdot \mathsf{P}_i + \nabla \cdot \mathsf{P}_e = \frac{\nabla \times \boldsymbol{B}}{4\pi} \times \boldsymbol{B} \tag{3}$$

$$\frac{d}{dt}\left[\frac{P_{\parallel}B^2}{N^3}\right] = 0 \tag{4}$$

$$\frac{d}{dt}\left[\frac{P_{\perp}}{NB}\right] = 0 \tag{5}$$

$$\frac{d}{dt}\left[\frac{p}{n^{5/3}}\right] = -\frac{1}{n^{5/3}}\frac{5}{3}\nabla \cdot (q\hat{e}) \tag{6}$$

$$\frac{d}{dt}\left[\frac{B}{n^{2/3}}\right] = \frac{B}{pn^{2/3}}\frac{1}{3}(q\nabla \cdot \hat{e} - 2\hat{e} \cdot \nabla q) \tag{7}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{8}$$

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \frac{\partial \boldsymbol{B}}{\partial t} \tag{9}$$

where

n: electron number density,
N: proton number density,
z: ion atomic number (1 in the present case),
M: ion mass,
$\boldsymbol{u}$: bulk velocity,
$\boldsymbol{B}$: magnetic field,
P_i: ion pressure tensor,
$\overleftrightarrow{\boldsymbol{P}}_e$: electron pressure tensor,
f: electron heat flux,
$\hat{e} = \boldsymbol{B}/B$
$q = \frac{2}{5}f$
$\mathsf{P}_i = P_{\parallel}\hat{e}\hat{e} + P_{\perp}(\mathsf{I} - \hat{e}\hat{e})$
$\mathsf{P}_e = p\,\mathsf{I}$
I = identy Tensor

The first equation is obtained from the quasi neutrality condition and the second shows the mass conservation. The equation of motion is (3), whereas (4) and (5) are related to the ions' energy (they are the same as the CGL energy equations). The new electrons' energy equations are (6) and (7), whereas (8) and (9) are Maxwell's equations, the last one for a fluid in the MHD approximation. Note that (3) couples the thermodynamic variables of electrons and ions, whereas (7) relates the heat flux to the magnetic field, which couples the electrons' and ions' energy equations (for further details on the model see appendix).

3. EQUIVALENT SET OF EQUATIONS OF CONSERVATION LAWS

The previous set of equations can be re-written in conservation form as

$$\frac{\partial N}{\partial t} + \nabla \cdot (N\boldsymbol{u}) = 0 \tag{10}$$

$$\frac{\partial}{\partial t}(MN\boldsymbol{u}) + \nabla \cdot \left(MN\boldsymbol{u}\boldsymbol{u} + \mathsf{P}_i + \mathsf{P}_e - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi} + \right.$$
$$\left. + \frac{B^2}{8\pi}\mathsf{I}\right) = 0 \tag{11}$$

$$\frac{\partial}{\partial t}\left[\frac{P_{\parallel}B^2}{N^2}\right] + \nabla \cdot \left[\frac{P_{\parallel}B^2}{N^2}\boldsymbol{u}\right] = 0 \tag{12}$$

$$\frac{\partial}{\partial t}\left[\frac{P_{\perp}}{B}\right] + \nabla \cdot \left[\frac{P_{\perp}}{B}\boldsymbol{u}\right] = 0 \tag{13}$$

$$\frac{\partial}{\partial t}\left(MN\frac{u^2}{2} + \frac{P_{\parallel}}{2} + P_{\perp} + \frac{3}{2}p + \frac{B^2}{8\pi}\right) +$$
$$+\nabla \cdot \left(\left[MN\frac{u^2}{2} + \frac{P_{\parallel}}{2} + P_{\perp} + \frac{3}{2}p\right]\boldsymbol{u} + \right.$$
$$\left. + (\mathsf{P}_i + \mathsf{P}_e) \cdot \boldsymbol{u} + \frac{-\boldsymbol{u} \times \boldsymbol{B}}{4\pi} \times \boldsymbol{B} + \boldsymbol{f}\right) = 0 \tag{14}$$

$$\frac{\partial}{\partial t}(Bn^{1/3}) + \nabla \cdot (Bn^{1/3}\boldsymbol{u}) =$$
$$\frac{Bn^{1/3}}{p}\frac{2}{15}(f\nabla \cdot \hat{e} - 2\hat{e} \cdot \nabla f) \tag{15}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{16}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \cdot (\boldsymbol{u}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{u}) = 0 \tag{17}$$

Note that only (15) has a source term. If a process were conservative, then $f = 0$ and all equations would be equal to zero.

The conservation form of a set of equations is very useful in various topics, as discontinuities (Lynn, 1967) for example. Our aim is to apply it in the study of the coronal expansion (in the collisionfree region). This allows us to obtain the conservation of various physical quantities.

4. CORONAL EXPANSION IN THE COLLISIONFREE ZONE

The coronal expansion is usually studied with the assumptions of spherical symmetry and stationary state (in the equatorial plane). Applying these conditions to the conservation set we obtain:

$$N u_r r^2 = C_1 \tag{18}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\left[M N u_r^2 + (P_\parallel - P_\perp)\frac{B_r^2}{B_r^2+B_\varphi^2} + P_\perp + p - \frac{B_r^2}{4\pi} + \frac{B_r^2+B_\varphi^2}{8\pi}\right]\right) + \frac{1}{r}\left[-M N u_\varphi^2 - (P_\parallel - P_\perp)\frac{B_\varphi^2}{B_r^2+B_\varphi^2} - 2p - 2P_\perp - \frac{B_r^2}{4\pi} + G\frac{M M_\theta N}{r^2}\right] = 0 \tag{19}$$

$$r^3\left(M N u_\varphi u_r + (P_\parallel - P_\perp)\frac{B_r B_\varphi}{B_r^2+B_\varphi^2} + -\frac{B_r B_\varphi}{4\pi})\right) = C_2 \tag{20}$$

$$r^2\frac{P_\perp}{(B_r^2+B_\varphi^2)^{1/2}}u_r = C_3 \tag{21}$$

$$r^2 P_\parallel\frac{(B_r^2+B_\varphi^2)}{N^2}u_r = C_4 \tag{22}$$

$$r^2\left\{\left[\frac{1}{2}M N(u_r^2+u_\varphi^2) + \frac{P_\parallel}{2}P_\perp + \frac{3}{2}p\right]u_r + \left(\frac{P_\parallel B_r^2 + P_\perp B_\varphi^2}{B_r^2+B_\varphi^2} + p\right)u_r + (P_\parallel - P_\perp)\frac{B_r B_\varphi}{B_r^2+B_\varphi^2}u_\varphi + \frac{B_\varphi^2}{4\pi}u_r - u_\varphi\frac{B_r B_\varphi}{4\pi} + f\frac{B_r}{(B_r^2+B_\varphi^2)^{1/2}} - M\,N\frac{G M_\theta r}{u_r}\right\} = C_5 \tag{23}$$

$$\frac{B_r}{B}\frac{dB}{dr}q = -u_r p^{7/5}\frac{d}{dr}\left[\frac{B}{p^{2/5}}\right] \tag{24}$$

$$r^2 B_r = C_6 \tag{25}$$

$$r(u_r B_\varphi - u_\varphi B_r) = C_7 \tag{26}$$

r: heliocentric distance,
φ: azimuthal angle,
G: gravitational constant,
M_θ: mass of the sun.

Notice that the gravitational force has been included now. C_1 through C_7 are constants (nondependent on the heliocentric distance), whereas only two equations do not lead to conservated quantities: the radial momentum equation and the second new electron equation.

From the combination of these equations it follows the conservation of:
mass rate:

$$\mathrm{M} = M N u_r r^2 \tag{27}$$

specific angular momentum:

$$L = r\left\{u_\varphi - \frac{B_r B_\varphi}{4\pi M N u_r}\left[1 - \frac{4\pi N}{B_r^2+B_\varphi^2}\left(\frac{(P_\parallel - P_\perp)}{N}\right)\right]\right\} \tag{28}$$

magnetic moment:

$$\mu = \frac{M P_\perp}{N(B_r^2+B_\varphi^2)^{1/2}} \tag{29}$$

second adiabatic invariant:

$$\lambda = M P_\parallel\frac{B_r^2+B_\varphi^2}{N^3} \tag{30}$$

total energy per proton mass:

$$\varepsilon = \frac{1}{2}(u_r^2 + u_\varphi^2) + \frac{P_\parallel/2 + P_\perp + \frac{3}{2}p}{M\,N} +$$

$$+\frac{1}{M\,N}\left(\frac{P_\parallel B_r^2 + P_\perp B_\varphi^2}{B_r^2 + B_\varphi^2} + p\right) +$$

$$+\frac{u_\varphi}{u_r}\frac{(P_\parallel - P_\perp)}{MN}\frac{B_r B_\varphi}{B_r^2 + B_\varphi^2} + \frac{B_\varphi^2}{4\pi M N} +$$

$$-\frac{u_\varphi B_r B_\varphi}{u_r 4\Pi M N} + f\frac{\frac{B_r}{(B_r^2+B_\varphi^2)^{1/2}}}{M N u_r} - G\frac{M_\theta}{r} \qquad (31)$$

magnetic flux :

$$F = r^2 B_r \qquad (32)$$

C_7 is not a new independent conservated quantity, because it is related to C_6 as (see e.g. Weber and Davis, 1967):

$$C_7 = -\Omega C_6 \qquad (33)$$

Ω: angular velocity of the sun.

5. CONCLUSIONS

The present approach leads us to the following considerations:

1) A set of 9 coupled differential equations may be reduced to two coupled differential and seven algebraic equations, due to the fact that they have been written in conservation form.

2) Every algebraic equation is related to a constant of motion, so the value of that physical quantity may be measured at the most convenient distance. Helios particle and magnetic field observations between 0.3 and 1 AU were used by Marsch and Richter (1984) to determine plasma parameters that characterize the solar wind and which yield observational constraints on theoretical fluid models for the coronal expansion. The mass rate, specific angular momentum, total energy per proton mass of the solar wind and the magnetic flux, expected to be conserved in a time-stationary flow with local spherical symmetry in the ecliptic plane, are actually found to be invariant within measurement uncertainties.

6. APPENDIX

The following equation of motion for the electrons, which coincides with the Ohm's laws for this plasma (see e.g. Grad 1967), has been derived by Duhau (1984):

$$\nabla \cdot \mathsf{P}_e = n\,e\left(\boldsymbol{E} + \frac{\boldsymbol{u}\times\boldsymbol{B}}{c}\right) + \frac{\boldsymbol{J}\times\boldsymbol{B}}{c} \qquad (34)$$

where e is the electron charge, $\boldsymbol{E}$ the electric field in the reference system moving with the bulk velocity $\boldsymbol{u}$, c the speed of light and $\boldsymbol{J}$ the electric current. From this equation it may be found that:

$$-\boldsymbol{J}_\perp = \frac{c}{B}\nabla\cdot\mathsf{P}_e\times\hat{e} - \frac{c\,n\,e}{B}\boldsymbol{E}\times\hat{e} + n\,e\,\boldsymbol{u}_\perp \qquad (35)$$

Regarding Maxwell 's equations, (8) and (9) are Gauss' law for the magnetic field and the Faraday-Henry law respectively. The Ampere-Maxwell law in the MHD approximation, where the displacement current is negligible, has been used to derive the right hand side of equation (3). To find $\boldsymbol{J}_\parallel$ we must combine the remaining of Maxwell's equations, Gauss' law for the electric field, with the continuity equation for ρ and $\boldsymbol{J}$, where ρ is the mass density. Taking into account that the displacement current is negligible it follows that,

$$\nabla\cdot\boldsymbol{J} = 0 \qquad (36)$$

Therefore

$$\frac{\partial J_\parallel}{\partial_\parallel} = -\nabla_\perp\cdot\boldsymbol{J}_\perp \qquad (37)$$

where $\partial/\partial_\parallel$ means to take the spatial derivative along $\boldsymbol{B}$.

Equations (A29,30) give $\boldsymbol{j}$ as a function of $\bar{E}$ and plasma parameters. There is not a linear relationship between both.

Notice also that (7) replaces the classical heat conduction law $\boldsymbol{f} = -K\nabla T$, $K \sim T^{5/2}$, which is used under the assumption that the plasma is collision dominated. This condition certainly breaks down in the solar wind from some heliocentric distance on. As required by the observations (see e.g. Feldman et al., 1978), the new equation does not imply any a priori relationship between the direction of the heat flux and the temperature gradient.

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Versão recebida em: 1989
Versão Revisada em: 17/03/92
Versão aceita em: 26/03/92
Editor associado: V.W.J.H. Kirchhoff