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INFINITE SERIES, LIMITS, RELATIONSHIPS AND ALIASING

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This paper presents a coherent approach to the evaluation of limits of certain infinite series and to the establishment of relationships which exist between such series. The approach is based on the connection which exists between the Discrete Fourier Transform and the phenomenon of aliasing as expressed by the Poisson sum formula. One of the limits which follows naturally from the method described in this paper is for the well known Leibnitz series which converges to π. The interpretation of certain infinite sums as aliased spectra allows many interesting relationships between series to be derived and some new results are presented which may be of interest to communications theory.

Key words: Infinite series; Limits; Aliasing; Poisson sum formula.

SÉRIES INFINITAS, LIMITES, RELACIONAMENTOS E SUAVIZAÇÕES - *Este trabalho apresenta uma abordagem coerente de avaliação dos limites de algumas séries infinitas, e o estabelecimento do relacionamento que existe entre estas séries. A abordagem é baseada na conexão entre a Transformada Discreta de Fourier e o processo de suavização, da maneira como é expresso pela equação de Poisson. Um dos limites que o método descrito neste trabalho apresenta de modo natural, é em direção a muito bem conhecida série de Leibnitz, que converge para π. A interpretação de algumas somatórias infinitas, na forma de espectros suavizados, permite muitos relacionamentos interessantes entre séries derivadas, e alguns resultados de interesse à teoria de comunicação são apresentados.*

Palavras-chave: *Séries infinitas; Limites; Suavizações; Equação de Poisson.*

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INTRODUCTION

The Shannon sampling theorem (Shannon, 1949), together with its various extensions and applications, superbly reviewed by Jerri (1977), has been of fundamental importance both in information and communications theory. A related theorem, expressed by the Poisson sum formula (Papoulis, 1977), is also of considerable interest and it is this theorem and in particular its relationship to infinite series which is explored in this paper. As often happens in mathematics and communications theory, a result is obtained in terms of an infinite sequence of numbers which is defined to be an infinite series. Of particular importance are the limits of infinite series, if these exist, and relationships between series such as products for example. An example of the application of infinite series in geophysics which has generated a considerable amount of interest is the caracterization of a wavefield in an inhomogeneous medium as a wavefield in a homogeneous reference medium, and a perturbation which takes the form of an infinite series called the Born series. Recently, Carvalho et al. (1991) presented a procedure to remove all free surface multiples from marine seismic data using a subseries of the full inverse series. Araujo et al. (1994) have developed a method to attack the more difficult problem of removing interbed multiples by using a more subtle selection of inverse subseries.

Many books exist which develop expressions for limits of series and show relationships between series (for example Knopp, 1931), and our aim is not particularly to develop new limits or expressions. Rather, we develop in this paper a coherent approach to the analysis of certain types of infinite series which is based on the phenomenon of aliasing and we present some new results.

As is well known, aliasing is caused by the sampling of a continuous function of time, $x(t)$, at intervals of time, Δt. It is the result of the addition from zero to the Nyquist frequency, to $X(f)$, the Fourier Transform of $x(t)$, of all the contributions of the $X(f)$ which are replicated at intervals of $1/\Delta t$ in the frequency domain (Bracewell, 1978). The fact that the fully aliased spectrum is equal to the Discrete Fourier Transform, DFT, of the sampled data is expressed by the Poisson sum formula (Papoulis, 1977) which has been used in the past to derive a variety of interesting results (see e.g. Papoulis, 1977). This formula in fact states that the infinite sum representing the aliased spectrum may be computed in only $N\log_2 N$ operations using the Fast Fourier Transform, FFT, algorithm.

In certain instances, illustrated below, the DFT may be evaluated in closed form and we may equate the ensuing expression to the infinite series which is the aliased spectrum. This procedure leads to the limits of series which we mentioned above. Further, it so happens, that for functions which possess closed form DFT expressions, the ratio of their DFT's may be shown to be a constant. This observation will lead us to the formulation of expressions which contain relationships between series and in particular the products of infinite series.

THEORY

The Poisson sum formula is generally written as (Papoulis, 1977)

$$\sum_k X(f+\frac{k}{\Delta t}) = \Delta t \sum_n x(n\Delta t)e^{-i2\pi f n\Delta t}, \qquad (1)$$

where $x(n\Delta t)$ is $x(t)$ sampled at intervals Δt and where $\sum_j$ implies that the summation index j varies from $-\infty$ to $+\infty$. Since we will be concerned in this paper with functions of time which are causal and may be discontinuous at the origin, Eq.(1) must be slightly modified. Specifically, since a causal time function may be denoted by $\hat{x}(t) = x(t)H(t)$, where $H(t)$ is the Heavyside step function and is defined to be equal to 1/2 at $t = 0$, Eq.(1) becomes

$$\sum_k X(f+\frac{k}{\Delta t}) = \Delta t \sum_n x(n\Delta t)e^{-i2\pi f n\Delta t} - $$
$$-\frac{1}{2}\Delta t x(0). \qquad (2)$$

We emphasize at this point that the interpretation of Eq.(2) is that the infinite sum which arises from the superposition of the displaced transforms is equal to the DFT of x(t) which we will represent by X_f and which, naturally, is evaluated at discrete frequencies.

Limits of Infinite Series

In this section we develop an approach to the evaluation of the limits of certain infinite series. In particular we will be concerned with power and trigonometric series.

Power series

We begin by considering a causal function impulsive at the origin, $x(t) = e^{-\alpha t}$ with transform $X(f) = 1/(\alpha + i2\pi f)$. Sampling $x(t)$ at intervals Δt and performing the DFT we obtain

$$X_f = \Delta t \sum_n e^{-\alpha n \Delta t_e - i2\pi f n \Delta t} - \frac{\Delta t}{2}. \qquad (3)$$

If we let $r = e^{-(\alpha + i2\pi f)\Delta t}$ and since $\sum_{n=0}^{\infty} r^n = 1/(1-r)$ we may write Eq.(3) as

$$X_f = \frac{\Delta t}{1-r} - \frac{\Delta t}{2} = \frac{\Delta t}{2}\left(\frac{1+r}{1-r}\right).$$

Equating the aliased sum to X_f given by the above expression we obtain

$$\sum_k \frac{1}{\left(\alpha + i2\pi\left(f + \frac{k}{\Delta t}\right)\right)} = \frac{\Delta t}{2}\left(\frac{1+r}{1-r}\right). \qquad (4)$$

Let us, for convenience, assume $\Delta t = 2\pi$. Letting $\beta = 2\pi f$ we obtain from Eq.(4) for $|\beta| \leq 1/2$

$$\sum_k \frac{\alpha - i(\beta + k)}{\alpha^2 + (\beta + k)^2} = \pi\left(\frac{1 + e^{-\alpha 2\pi} e^{-i2\pi\beta}}{1 - e^{-\alpha 2\pi} e^{-i2\pi\beta}}\right). \qquad (5)$$

Splitting Eq.(5) into real and imaginary parts we derive the two following expressions

$$\sum_k \frac{\alpha}{\alpha^2 (\beta + k)^2} = \pi\left(\frac{1 - e^{-\alpha 4\pi}}{1 + e^{-\alpha 4\pi} - 2e^{-\alpha 2\pi}\cos 2\pi\beta}\right), \qquad (6)$$

and

$$\sum_k \frac{\beta + k}{\alpha^2 + (\beta + k)^2} = \pi\left(\frac{2e^{-\alpha 2\pi} sin 2\pi\beta}{1 + e^{-\alpha 4\pi} - 2e^{-\alpha 2\pi}\cos 2\pi\beta}\right) \qquad (7)$$

As particular examples, using Eq.(6) and substituting $\alpha = 1/(2\pi)$ and $\beta = 0$ we derive the limit

$$\frac{2\sum_k \frac{1}{1 + (2\pi k)^2} + 1}{2\sum_k \frac{1}{1 + (2\pi k)^2} - 1} = e$$

If we express Eq.(7) for the case $\alpha = 0$ and $\beta = 1/4$ we obtain the infinite series

$$\sum_k \frac{1}{\frac{1}{4} + k} = \pi. \qquad (8)$$

Evaluating the series in Eq.(8) for $k = -\infty$ to $+\infty$ we obtain

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right),$$

which is the well known Leibniz formula (Thomas & Finney, 1979).

Before proceeding, we must justify setting $\alpha = 0$ in Eq.(7). This appears to be incorrect since, as we know, the Fourier transform of $H(t)$ is equal to $1/2\delta(f) - i/(2\pi f)$ and not $-i/(2\pi f)$ as implied by setting $\alpha = 0$ in $X(f) = 1/(\alpha + i2\pi f)$. It turns out in fact that all the equations which we derive in this paper using Eq.(2), which implies a time series of infinite length, are valid for time limited functions. In other words, given

$$x(t) = \begin{cases} e^{-\alpha t} & 0 \leq t \leq N\Delta t \\ 0 & -\infty < t < 0;\ N\Delta t < t < \infty, \end{cases}$$

we show in the Appendix that Eq.(5) is obtained independently of N and consequently for any α.

At this stage we would like to mention a fact which may be of interest from a number theoretic view point. Let $e(L)$ represent the error between the value of π as computed using Eq.(8) for $k = -L$ to $+L$ and a "true" value of π computed using $\pi = \cos^{-1}(-1)$. We computed $e(L)$ using extended precision on a VAX 11/785 and obtained the following interesting results.

$$e(4999) = 1.0001000075002499 \times 10^{-4}$$
$$e(49999) = 1.0000100000750002499 \times 10^{-5}$$
$$e(499999) = 1.0000010000007500002499 \times 10^{-6}$$

The error exhibits a remarkably well defined structure. In fact, using this property, one may evaluate π using Eq.(8) and $L = 49999$, say, whith an accuracy of one part in 10^{23} rather than one part in 10^5. We remark that $e(L)$, which is proportional to $1/(2L+1)$, exhibits a similar, but different, structure which depends on the particular value of L.

We now look at a second example of determining limits of infinite series by considering the function $x(t) = te^{-\alpha t}$ for which $X(f) = 1/(\alpha + i2\pi)^2$. Using an identical approach and again letting $\Delta t = 2\pi$ for convenience we obtain

$$\sum_k \frac{1}{(\alpha + i(\beta + k))^2} = (2\pi)^2 \sum_n nr^n = $$
$$= (2\pi)^2 \frac{r}{(1-r)^2}.$$

Consequently we have

$$\sum_k \frac{1}{(\alpha + i(\beta + k))^2} = \frac{4\pi^2 e^{-2\pi\alpha} e^{-i2\pi\beta}}{(1 - e^{-2\pi\alpha} e^{-i2\pi\beta})^2}, \qquad (9)$$

again, evaluating Eq.(9) for particular cases, we determine that for $\alpha = 0$ and $\beta = 1/2$

$$\sum_k \frac{1}{\left(\frac{1}{2} + k\right)^2} = \pi^2.$$

For $\alpha = 0$ and $\beta = 1/4$ we obtain a similar result

$$\sum_k \frac{1}{\left(\frac{1}{4} + k\right)^2} = \frac{\pi^2}{2}.$$

Many other formulations are clearly possible and we leave the discussion of these formulations for later.

Trigonometric series

We considerar a continuous function of time $z(t)$ to which we add another windowed series $x(t)w(t/T)$ to obtain $y(t) = z(t) + x(t)w(t/T)$, where $w(t/T)$ is a window function of length T such that $w(t) = 0$ for $|t| \geq T/2$. We now sample $y(t)$ at intervals Δt where $\Delta t > T$ and T lies within one of the intervals Δt. It is clear that whereas $Y(f)$, the Fourier transform of $y(t)$, contains the spectral contribution of $x(t)w(t/T)$, the DFT of the sampled function, Y_p, which is the alised spectrum of $y(t)$, contains no information about $x(t)$. In other words, aliasing has annihilated all spectral contribution of $x(t)$. The very best that we can possibly expect to recover from the aliased spectrum is the function $z(t)$ itself. We now express this fact in a general form.

Let us window $x(t)$ with $w((t - n\Delta t - \tau)/T)$ where τ is the time shift of the midpoint of the window $w(t/T)$ with respect to the $(n+1)^{\text{th}}$ Δt interval and $0 < \tau < t$. T and τ are related through $T \leq 2\min(\tau, \Delta t - \tau)$ for all windows except for the boxcar window for which $T < 2\min(\tau, \Delta t - \tau)$. The fact that the aliased spectrum of $x(t)w((t - n\Delta t - \tau)/T)$ is zero can now be expressed by

$$\sum_k \int_{-\infty}^{+\infty} X(u) W(T(f + \frac{k}{\Delta t} - u)) e^{-i2\pi(n\Delta t + r)(f + k/\Delta t - u)} du = 0$$
$$= 0. \qquad (10)$$

We emphasize that Eq.(10) is true for any $X(f)$ whatsoever and for any window function W(f). As simple examples of series which arise from Eq.(10) we let $x(t) = 1$, $\Delta t = 1$, $n = 0$ and consider two window functions. For a triangular window function with $T = 1$, $\tau = 1/2$ and $f = 0$ we obtain

$$\sum_k \cos 2\pi k \frac{\sin^2 \frac{\pi}{2} k}{k^2} = 0,$$

(where for $k = 0$ the corresponding term is equal to $\pi^2/4$). For a boxcar window function, letting $T = 1/2$, we derive the series

$$\sum_k \cos 2\pi\tau(\beta + k) \frac{\sin \frac{\pi}{2}(\beta + k)}{(\beta + k)} = 0,$$

for $|\beta| \leq 1/2$ and $1/4 < |\tau| < 3/4$. This is and interesting series in that not only does it converge to zero but, for certain values of τ which depend on the number of terms summed, (and which are in fact the roots of the corresponding equation), the sum is actually zero.

Relationships between infinite series

In this section we develop a general approach which, for certain series, leads to interesting relationships between infinite sums. Let us define $z(t)$ to be the convolution of two continuous functions of time, $x(t)$ and $y(t)$, i.e. $z(t) = x(t) * y(t)$ (where * implies convolution). For certain functions, as we show below, the ratio expressed by

$$\frac{Z_f}{X_f} - Y_f = K_f, \qquad (11)$$

may be evaluated in closed form. K_f is related to the DFT of the difference between the analytic convolution, $z(t) = x(t) * y(t)$, and the discrete convolution, $z(n\Delta t) = \sum_m x(m\Delta t)y((n-m)\Delta t)$, and in certain instances is a constant. Writing Eq.(11) in terms of the series which correspond to the aliased spectra of $x(t)$,$y(t)$ and $z(t)$ we obtain expressions which contain the products of infinite series. As an example of this approach we consider $x(t) = y(t) = e^{-\alpha t}H(t)$ and $z(t) = te^{-\alpha t}H(t)$. Using the same development as before where $r = e^{-(\alpha+i2\pi f)\Delta t}$, we form

$$X_f = \frac{\Delta t}{2}\left(\frac{1+r}{1-r}\right),\ Z_f = \Delta t^2 \frac{r}{(1-r)^2}.$$

Because of the discontinuous behaviour of $x(t)$ at $t = 0$ we modify Eq.(11) slightly and using the above expressions we form

$$\frac{Z_f}{(X_f + \frac{1}{2}\Delta t)} - \left(X_f + \frac{1}{2}\Delta t\right) = -\Delta t. \qquad (12)$$

Putting Eq.(12) into the form of infinite series and assuming for convenience as before that $\Delta t = 2\pi$ we obtain for $|\beta| \le 1/2$

$$\sum_k \frac{(\alpha - i(\beta+k))^2}{(\alpha^2 + (\beta+k)^2)^2} = \left(\sum_k \frac{a - i(\beta+k)}{\alpha^2 + (\beta+k)^2}\right)^2 - \pi^2$$

Equating real and imaginary parts

$$\sum_k \frac{\alpha^2 - (\beta+k)^2}{(\alpha^2 + (\beta+k)^2)^2} =$$
$$= \left(\sum_k \frac{\alpha}{\alpha^2 + (\beta+k)^2}\right)^2 - \left(\sum_k \frac{\beta+k}{\alpha^2 + (\beta+k)^2}\right)^2 - \pi^2$$

$$\sum_k \frac{\alpha(\beta+k)}{(\alpha^2 + (\beta+k)^2)^2} =$$
$$= \sum_k \frac{\alpha}{\alpha^2 + (\beta+k)^2} \sum_k \frac{\beta+k}{\alpha^2 + (\beta+k)^2}. \qquad (13)$$

In passing we remark that particular cases follow immediately from the interpretation of the sums in terms of aliasing. Thus, since aliasing of the imaginary part of the DFT is zero at the Nyquist frequency, $\beta = 1/2$, it follows immediately from Eq.(13) that

$$\sum_k \frac{\alpha(\frac{1}{2}+k)}{(\alpha^2 + (\frac{1}{2}+k)^2)^2} =$$
$$= \sum_k \frac{\alpha}{\alpha^2 + (\frac{1}{2}+k)^2} \sum_k \frac{\frac{1}{2}+k}{\alpha^2 + (\frac{1}{2}+k)^2} = 0.$$

We mention, without actually evaluating the complete expressions, two other possibilities for generating products of series

Case I.

If $x(t) = te^{-\alpha t}$, $y(t) = e^{-\alpha t}$ and $z(t) = x(t) * y(t)$ then

$$\frac{Z_f}{X_f} - Y_f = 0.$$

This leads to products of the form

$$\sum_k \frac{1}{(\alpha + i(\beta+k))^3} =$$
$$= \sum_k \frac{1}{(\alpha + i(\beta+k))^2} \sum_k \frac{1}{(\alpha + i(\beta+k))}.$$

Case II.

We let $x(t) = y(t) = te^{-\alpha t}$. Then

$$\frac{Z_f}{X_f} - Y_f = \frac{\Delta t^2}{6}.$$

This expression allows the development of relationships between sums where the left hand side of the equations has the form

$$\sum_k \frac{1}{(\alpha + i(\beta+k))^4}.$$

DISCUSSION

We have presented a new method of determining limits of certain types of infinite sums which stems from the relationship of aliasing to the DFT as expressed by the Poisson sum formula. The observation that the DFT's of certain functions are related very simply when expressed as in Eq.(11) has led to the development of various product formulae and to various expressions relating infinite sums.

Our method depends on obtaining a closed form expression for the DFT and is very rich in possibilities. Specifically, if we let $S_j = \sum_{n=0}^{\infty} n^j r^n$, we can establish the recursion $S_j = r\frac{d}{dr}S_{j-1}$ for $j = 1,2,\ldots$, compute the required expressions for the DFT's of functions such as $x(t) = t^n e^{-\alpha t}$ and thereby generate a multitude of limits and relationships between infinite sums. In this paper we have only indicated, with examples, the general approach which is to be followed.

Our approach is, of course, valid for functions which are not necessarily causal. For example, computing the closed form DFT of the equivalent filter in the frequency domain which produces linear interpolation in the time domain and using Eq.(1) yields the relationship

$$\sum_k \frac{\sin^2 2\pi(\beta+k)}{(2\pi(\beta+k))^2} = \cos^2 \pi\beta, \quad |\beta| \le \frac{1}{2}.$$

Finally we would like to emphasize the role which the DFT can play in the evaluation of infinite sums. Let us assume that we wish to compute the value of $\sum_k Q(u+k)$ to some given accuracy. If $Q(u)$ may be expressed as the Fourier transform of some continuous function of time, $q(t)$, then the limit may be evaluated in $N\log_2 N$ operations using the FFT algorithm to an accuracy allowed for by the particular computer. For example, suppose we wish to determine $\sum_k \tan^{-1} 1/(2\pi(\beta+k))^2$ for $|\beta| \le 1/2$ to an accuracy of one part in 10^7. Since $\tan^{-1} 1/(2\pi f)^2$ is the Fourier transform of $x(t) = e^{-|t|} \sin t/t$ the infinite sum may be evaluated by discretising $x(t)$ at unit time intervals and evaluating the FFT at the particular value of β. Since an accuracy of one part in 10^7 requires the summation index k to run between -10^6 and $+10^6$ the ratio of the times required for the computation using the sum itself and the FFT is 10^3. Of course another advantage of the FFT approach, where it is in fact an option, is that the FFT at the same time evaluates the sum at all other values of β depending only on the number of points used in the FFT.

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APPENDIX

We show here that Eq.(5) in the text is valid for time-limited functions and is independent of the length of the time function. This in turn implies that Eq.(5) is valid for any value of α, including zero.

Consider the following time-limited, causal function

$$x(t) = \begin{cases} e^{-\alpha t} & 0 \le t \le N\Delta t \\ 0 & -\infty < t < 0 \; ; N\Delta t < t < \infty \end{cases}$$

The Fourier transform is given by

$$X(f) = \int_0^{N\Delta t} e^{-\alpha t} e^{-i2\pi f t} dt = \left. \frac{e^{-\alpha t - i2\pi f t}}{\alpha + i2\pi f} \right|_0^{N\Delta t} =$$

$$= \frac{1}{\alpha + i2\pi f} - \frac{e^{-\alpha N\Delta t - i2\pi f N\Delta t}}{\alpha + i2\pi f}.$$

The corresponding aliased spectrum, $X^a(f)$, is

$$X^a(f) = \sum_k \frac{1}{\alpha + i2\pi(f + \frac{k}{\Delta t})} - $$
$$-\sum_k \frac{e^{-\alpha N\Delta t - i2\pi(f + k/\Delta t)N\Delta t}}{\alpha + i2\pi(f + \frac{k}{\Delta t})} =$$
$$= \left[1 - e^{-(\alpha + i2\pi f)N\Delta t}\right] \sum_k \frac{1}{\alpha + i2\pi(f + \frac{k}{\Delta t})}. \quad \text{(A-1)}$$

On the other hand, computing the DFT, and using the fact that

$$\sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r},$$

we have

$$\Delta t \sum_n x(n\Delta t) e^{-i2\pi f n\Delta t} =$$
$$= \Delta t \sum_{n=0}^{N-1} e^{-(\alpha + i2\pi f)n\Delta t} - \frac{\Delta t}{2} + \frac{\Delta t}{2} e^{-(\alpha + i2\pi f)N\Delta t}$$
$$= \frac{\Delta t}{2} \left(\frac{1 + e^{-(\alpha + i2\pi f)\Delta t}}{1 - e^{-(\alpha + i2\pi f)\Delta t}} \right) \left[1 - e^{-(\alpha + i2\pi f)N\Delta t}\right]. \quad \text{(A-2)}$$

Now, substituting Eq.(A-1) and Eq.(A-2) into Eq.(2), we obtain

$$\left(1 - e^{-(\alpha + i2\pi f)N\Delta t}\right) \left[\sum_k \frac{1}{\alpha + i2\pi(f + \frac{k}{\Delta t})} - \frac{\Delta t}{2} \left(\frac{1 + e^{-(\alpha + i2\pi f)\Delta t}}{1 - e^{-(\alpha + i2\pi f)\Delta t}} \right) \right] = 0. \quad \text{(A-3)}$$

From Eq.(A-3) we conclude that

$$\sum_k \frac{1}{\alpha + i2\pi(f + \frac{k}{\Delta t})} = \frac{\Delta t}{2} \left(\frac{1 + e^{-(\alpha + i2\pi f)\Delta t}}{1 - e^{-(\alpha + i2\pi f)\Delta t}} \right), \quad \text{(A-4)}$$

an expression which is independent of N and is valid for any Δt and α. Eq.(A-4) is equivalent to Eq.(5) and thus we have established the justification of substituting $\alpha = 0$ in Eq.(7).

The above approach is quite general. For example, considering

$$x(t) = \begin{cases} te^{-\alpha t} & 0 \le t \le N\Delta t \\ 0 & -\infty < t < 0;\ N\Delta t < t < \infty \end{cases}$$

using the fact that

$$\sum_{n=0}^{N-1} nr^n = \frac{r(1 - r^N)}{(1 - r)^2} - \frac{Nr^N}{(1 - r)},$$

and using the above development we can establish that

$$\left(1 - e^{-(\alpha + i2\pi f)N\Delta t}\right) \left[\sum_k \frac{1}{\left(\alpha + i2\pi(f + \frac{k}{\Delta t})\right)^2} - \Delta t^2 \frac{e^{-(\alpha + i2\pi f)\Delta t}}{\left(1 - e^{-(\alpha + i2\pi f)\Delta t}\right)^2} \right] = 0.$$

The term in N once again factors out of the equation and we obtain

$$\sum_k \frac{1}{\left(\alpha + i2\pi(f + \frac{k}{\Delta t})\right)^2} = \Delta t^2 \frac{e^{-(\alpha + i2\pi f)\Delta t}}{\left(1 - e^{-(\alpha + i2\pi f)\Delta t}\right)^2}. \quad \text{(A-5)}$$

an expression independent of N and valid for all Δt and α. Eq.(A-5), with $\Delta t = 2\pi$, is equivalent to Eq.(9) which was derived from Eq.(2) with far less algebra.

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