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# ON THE TIDAL TIME DELAY OF THE EARTH

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We present an expression for the tidal time delay of the Earth, that takes into account paleonthological observations. A semi-empirical model of the dynamics of the Earth-Moon system is obtained considering that the energy dissipation and the time delay are not constant in time. In a previous work, we established a similar model by taking into account only the dynamics of a circular orbit radius for the Moon. In the present paper, three differential equations describe the dynamics of three variables, namely, the Moon's orbital mean radius, the eccentricity of the orbit, and the inclination of the Moon's orbital plane with respect to the inertial plane which is orthogonal to the total angular momentum of the Earth-Moon system. In this sense, the Earth-Moon system is considered isolated. We compare our picture with that one which considers the tidal time delay as constant in time and show that our approach is consistent with the modern theories concerning the Earth-Moon system formation.

**Key words**: Tidal time delay; Earth-Moon system; Dynamic systems.

**SOBRE O ATRASO DE RESPOSTA DE MARÉS DA TERRA** - *Apresentamos uma expressão para o atraso de resposta das marés da Terra, a qual leva em consideração observações paleontológicas. Um modelo semi-empírico para a dinâmica do sistema Terra-Lua é obtido, considerando que a dissipação de energia e o atraso de resposta não são constantes em relação ao tempo. Em um trabalho anterior estabelecemos um modelo similar considerando apenas a dinâmica do raio de uma órbita circular da Lua. No presente artigo três equações diferenciais descrevem a dinâmica de três variáveis, a saber, o raio médio da órbita lunar, a excentricidade da órbita e a inclinação do plano orbital da Lua em relação ao plano inercial ortogonal ao momento angular total do sistema Terra-Lua. Neste sentido, consideramos o sistema Terra-Lua isolado. Comparamos nossa descrição com aquela que considera o atraso de resposta de maré constante em relação ao tempo e mostramos que nosso modelo é consistente com as modernas teorias sobre a formação do sistema Terra-Lua.*

***Palavras-chave:*** *Atraso de resposta das marés; Sistema Terra-Lua; Sistemas dinâmicos.*

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## INTRODUCTION

Since ancient observations of eclipses, astronomers concluded that there was a secular acceleration of the Moon. E. Halley (1695) tried to explain this phenomenon relating it to the gravitational perturbation of other planets. In 1787, P.S. Laplace announced that he had an explanation in terms of the action of the Sun on the lunar orbit. Nevertheless, in 1853, J. Adams showed that Laplace's model was not in agreement with the observations (Mignard, 1980). The philosopher E. Kant, in 1754, was the first one to suggest that the tides should be the cause of the lengthening of the day and of the secular acceleration of the Moon. It is interesting to remark that Kant's model appeared before Laplace's explanation for the acceleration of the Moon.

Several authors have given a global treatment to the effects of the tides on the Moon's orbit and on the Earth's rotation. Some of them are P. Goldreich (1966), W. M. Kaula (1964), G. MacDonald (1964), and F. Mignard (1979).

We combine G. Darwin's formalism of a tidal time delay (Goldreich & Soter, 1966) with a differential equation that allows us to estimate the dynamics of the rotation of our planet. The time delay corresponds to a time interval $\Delta t$ between the perturbation in the Earth due to the Moon and the instant when the tide is formed as an effect of the perturbation. In the calculations of the perturbation caused on the Earth-Moon system by tides, we use also the Love numbers formalism (Love, 1944). One of the differential equations that we have used is empirically obtained from paleonthological observations. The intervals of time that regulate the Earth's life are the tropical year, the synodical month, and the solar day. Hence, paleonthology appears as an independent key to estimate these time intervals. J.W. Wells (1963) suggested that some vital cycles of corals of the Middle Devonian may be identified. So, it is possible to estimate that, e.g., a tropical year had 400 solar days, 370 million years ago. With these data, it is possible to estimate the duration of the day and even the distance between the Earth and the Moon, hundreds of millions of years ago (Afonso, 1977).

We show the compatibility between our model and modern techniques with laser to measure the lunar motion. And we make a comparison, also, between our model and recent theories about the planetary system formation in our solar system.

In a previous work we established a similar model by taking into account only the dynamics of the Moon's orbit radius (the orbit was considered as circular) (Afonso & Sant'Anna, 1989; Sant'Anna, 1989). In the present paper, three differential equations are used to describe the dynamics of three variables, namely, the Moon's orbital mean radius, the eccentricity of the orbit, and the inclination of the Moon's orbital plane with respect to the inertial plane which is orthogonal to the total angular momentum of the Earth-Moon system. In this sense the Earth-Moon system is considered as isolated.

Our picture may be extended for any planet-satellite system where the interaction with other celestial bodies may be ignored.

This paper has four Sections. In Section 2 we present the dynamic equations for the Earth-Moon system as F. Mignard (1980) does it, i.e., by considering the time delay as a constant with respect to time. In Section 3 we develop a semi-empirical model based on paleonthological observations. By comparing the two models we obtain a non-trivial expression for the time delay as a function of time. Finally, in the last Section, we present some conclusions and remarks.

## DYNAMIC EQUATIONS WITH A CONSTANT TIME DELAY

The perturbative potential caused by tides on the Earth, considering an instantaneous response (time delay $\Delta t$ is zero), as a first approach, is given, according to Mignard (1979), by

$$U(\mathbf{r}) = \sum_{l=2}^{\infty} k_l \frac{Gm^* R_E^{2l+1}}{r^{l+1} r^{*l+1}} P_l \left[ \frac{\mathbf{r} \cdot \mathbf{r}^*}{r r^*} \right] \tag{1}$$

where $k_l$ denotes the Love number of order $l$, $G$ is the gravitational constant, $m^*$ is the mass of the Moon, $R_E$ is the equatorial radius of the Earth, $\mathbf{r}^*$ is the position vector of the gravitational center of the Moon with respect to the gravitational center of our planet, $r^*$ is the absolute value of $\mathbf{r}^*$, and $P_l$ is the Legendre polynomial of order $l$.

In order to take into account the time delay $\Delta t$ in our calculations, we have to perform a variable transformation on $\mathbf{r}$ and $\mathbf{r}^*$, considering the Moon's translation and the Earth's rotation. Hence, our new variables are:

$$\mathbf{r}_1 = \mathbf{r} \tag{2}$$

and

$$\mathbf{r}^*_1 = \mathbf{r}^* \,|_{\,t-\Delta t} + \Delta t\omega \times \mathbf{r}^* = \mathbf{r}^* - \mathbf{v}^*\Delta t + \omega \times \mathbf{r}^*\Delta t, \tag{3}$$

where $\mathbf{v}^*$ is the velocity of the translation of the Moon and $\omega$ corresponds to the angular velocity of the Earth. So, the new expression for the potential is:

$$U\,(\mathbf{r}_1, \mathbf{r}_1^*) = \sum_2^{\infty} U_l\,(\mathbf{r}_1, \mathbf{r}_1^*) \tag{4}$$

where

$$U_l\,(\mathbf{r}_1, \mathbf{r}_1^*) = k_l G m^* R_E^{2l+l}\ \varphi_l\,(r_1, r_1^*)\ P_l(x), \tag{5}$$

$$x = \frac{\mathbf{r}_1 \cdot \mathbf{r}_1^*}{r_1 r_1^*}, \tag{6}$$

and

$$\varphi_l\,(r_1, r_1^*) = \frac{1}{r_1^{l+1}\ r_1^{*l+1}}. \tag{7}$$

If we expand $U_l$ as a Taylor series, the term of first order with respect to $\Delta t$ is

$$\nabla_{\mathbf{r}_1^*} U_l \Big|_{\mathbf{r}_1^* = \mathbf{r}^*} (\mathbf{r}_1^* - \mathbf{r}^*), \tag{8}$$

where the first factor of the expression given above may be written as

$$\nabla_{\mathbf{r}_1^*} U_l \Big|_{\mathbf{r}_1^* = \mathbf{r}^*} = k_l G m^* R_E^{2l+1} \left[ P_l(x) \frac{\partial}{\partial r^*} \varphi_l\,(r, r^*) \frac{\mathbf{r}^*}{r^*} + \right.$$
$$\left. + \varphi_l\,(r, r^*) \frac{d}{dx} P_l\,(x) \left( \frac{\mathbf{r}}{rr^*} - \frac{(\mathbf{r} \cdot \mathbf{r}^*)\,\mathbf{r}^*}{r\,r^{*3}} \right) \right]. \tag{9}$$

So, each term of the expanded potential may be written as

$$V_l = k_l\, G m^* \Delta t R_E^{2l+1} \left[ (\mathbf{r}^* \cdot \mathbf{v}^*) \left( \frac{(\mathbf{r} \cdot \mathbf{r}^*)\,\varphi_l\, P_l'(x)}{r r^{*3}} - \frac{P_l(x)}{r^*} \frac{\partial}{\partial r^*} \varphi_l \right) + \right.$$
$$\left. + \frac{\mathbf{r} \cdot (\boldsymbol{\omega} \times \mathbf{r}^* - \mathbf{v}^*)}{r\,r^*} \varphi_l\, P_l'\,(x) \right] \tag{10}$$

where

$$P_l' = \frac{d}{dx} P_l\,(x). \tag{11}$$

In the Taylor series, the term which does not depend on $\Delta t$ is neglected because we are interested on the dissipative effects. The terms of superior order with respect to $\Delta t$ are ignored because we consider $\Delta t$ sufficiently small.

Now, we perform the calculation of the perturbative force:

$$\mathbf{F} = \sum_2^{\infty} \mathbf{F}_l, \tag{12}$$

where

$$\mathbf{F}_l = m^* \nabla_r V_l. \tag{13}$$

Since we are interested on the secular perturbations exerted on the Moon itself, we consider $\mathbf{r}^* = \mathbf{r}$ and $m^* = m$. Hence

$$\mathbf{F}_l = -k_l G m^2 \Delta t R_E^{2l+1} \left[ \frac{\mathbf{r}\,(\mathbf{r} \cdot \mathbf{v})}{r^2} \left( \frac{(l+1)^2}{r^{2l+4}} - \frac{1}{r^{2l+4}} P_l'\,(1) \right) + \right.$$
$$\left. + \frac{1}{r^{2l+2}} P_l'\,(1) \left( \frac{\mathbf{r} \times \omega}{r^2} + \frac{\mathbf{v}}{r^2} \right) \right]. \tag{14}$$

Recalling that $P_l'\,(1) = l\,(l+1)/2$

$$\mathbf{F}_l = -G m^2 k_l \Delta t R_E^{2l+1} \frac{1}{r^{2(l+3)}} \left[ \frac{(l+1)\,(l+2)}{2} (\mathbf{r} \cdot \mathbf{v})\,\mathbf{r} + \right.$$
$$\left. + \frac{l\,(l+1)}{2} r^2\,(\mathbf{v} + \mathbf{r} \times \boldsymbol{\omega}) \right] \tag{15}$$

The expression for the perturbative torque is obtained as follows:

$$\mathbf{T} = \sum_2^{\infty} \mathbf{T}_l, \tag{16}$$

where

$$\mathbf{T}_l = \mathbf{r} \times \mathbf{F}_l. \tag{17}$$

Hence:

$$\mathbf{T}_l = -Gm^2 k_l \Delta t R_E^{2l+1} \frac{1}{r^{2(l+1)}} \left[ \frac{l(l+1)}{2} ((\mathbf{r}\cdot\omega)\,\mathbf{r} - r^2\omega + \mathbf{r}\times\mathbf{v}) \right] \quad (18)$$

since $\mathbf{r}\times(\mathbf{r}\times\omega) = (\mathbf{r}\cdot\omega)\mathbf{r} - (\mathbf{r}\cdot\mathbf{r})\omega$.

Now, in order to obtain the dynamic equations of the Earth-Moon system, we use the Gaussian formalism for perturbations. Nevertheless, to do that, we need to consider $R$, $S$, and $W$ as three components (mutually perpendicular) of the perturbative acceleration $\mathbf{F}/\mu$, where $\mu$ is the reduced mass of the physical system that we are considering. These three components are given by

$$R = \sum_{2}^{\infty} R_l, \quad (19)$$

$$S = \sum_{2}^{\infty} S_l, \quad (20)$$

$$W = \sum_{2}^{\infty} W_l, \quad (21)$$

where

$$R_l = \frac{\mathbf{F}_l\cdot\mathbf{r}}{\mu r}, \quad (22)$$

$$S_l = \frac{\mathbf{F}_l\cdot(\mathbf{H}_M\times\mathbf{r})}{\mu|\mathbf{H}_M\times\mathbf{r}|}, \quad (23)$$

and

$$W_l = \frac{\mathbf{F}\cdot\mathbf{H}_M}{\mu H_M}, \quad (24)$$

where

$$\mathbf{H}_M = m\mathbf{r}\times\mathbf{v}, \quad (25)$$

and

$$H_M = mna^2(1-e^2)^{1/2}, \quad (26)$$

with

$$n = \left(\frac{GM}{a^3}\right)^{1/2}, \quad (27)$$

where $a$ is the orbit's mean radius, $e$ is the eccentricity of the orbit. $\mathbf{H}_M$ is the angular momentum of the Moon and $H_M$ is the norm of the vector $\mathbf{H}_M$. Eq. (27) corresponds to Kepler's Third Law.

Consider, now, the following relations:

$$\mathbf{v}\cdot(\mathbf{H}_M\times\mathbf{r}) = m(\mathbf{v}\cdot((\mathbf{r}\times\mathbf{v})\times\mathbf{r}) = m((\mathbf{r}\times\mathbf{v})\cdot(\mathbf{r}\times\mathbf{v}))$$

which means that

$$\mathbf{v}\cdot(\mathbf{H}_M\times\mathbf{r}) = mn^2a^4(1-e^2), \quad (28)$$

$$(\mathbf{H}_M\times\mathbf{r})\cdot(\mathbf{r}\times\omega) = m((\mathbf{r}\times\mathbf{v})\times\mathbf{r})\cdot(\mathbf{r}\times\omega) =$$
$$= m(\omega\cdot(((\mathbf{r}\times\mathbf{v})\times\mathbf{r})\times\mathbf{r})),$$

which means that

$$(\mathbf{H}_M\times\mathbf{r})\cdot(\mathbf{r}\times\omega) = -mr^2na^2(1-e^2)^{1/2}\,\omega\cos I, \quad (29)$$

and

$$\mathbf{H}_M\cdot(\mathbf{r}\times\omega) = m(\mathbf{r}\times\mathbf{v})\cdot(\mathbf{r}\times\omega) = m\mathbf{r}\cdot(\omega\times(\mathbf{r}\times\mathbf{v})) =$$
$$= mr\,|\omega\times(\mathbf{r}\times\mathbf{v})|\cos(\varpi+\nu)\sin I$$

which means that

$$\mathbf{H}_M\cdot(\mathbf{r}\times\omega) = mrna^2(1-e^2)^{1/2}\,\omega\cos(\varpi+\nu)\sin I, \quad (30)$$

where $I$ is the angle between the Moon's orbital plane and the equatorial plane of the Earth, $\varpi$ is the longitude of the perigee measured from the ascendent node and n is the true anomaly of the satellite.

Now, we may develop the expressions for $R_l$, $S_l$ and $W_l$:

$$R_l = \frac{-Gm^2k_l\Delta tR_E^{2l+1}}{\mu r}\frac{1}{r^{2(l+3)}}\left[\frac{(l+1)(l+2)}{2}(\mathbf{r}\cdot\mathbf{v})\mathbf{r}\cdot\mathbf{r} + \frac{l(l+1)}{2}r^2(\mathbf{r}\cdot\mathbf{v})\right],$$

that is,

$$R_l = \frac{-Gm^2k_l\Delta tR_E^{2l+1}}{\mu r^{2l+5}}(\mathbf{r}\cdot\mathbf{v})(l+1)^2 \quad (31)$$

But $\mathbf{r} = r\hat{r}$, and $\mathbf{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\sigma}$, where the dot denotes the derivative with respect to time. The versor $\hat{r}$ is perpendicular to the versor $\hat{\sigma}$, where this last one is parallel to the orbital plane. $\dot{\theta}$ corresponds to the angular velocity of the Moon (translation motion). Hence

$$R_l = -G\frac{m^2}{\mu}k_l \Delta t R_E^{2l+1}(l+1)^2 \frac{ne}{a^{2l+3}(1-e^2)^{1/2}}\left(\frac{a}{r}\right)^{2(l+2)} \sin\nu, \tag{32}$$

since

$$\mathbf{r}\cdot\mathbf{v} = r\dot{r} = \frac{naer\sin\nu}{(1-e^2)^{1/2}}. \tag{33}$$

The expression for $S_l$ is obtained as follows:

$$S_l = \frac{Gm^2 k_l \Delta t R_E^{2l+1}}{\mu r^{2(l+3)}|\mathbf{H}_M \times \mathbf{r}|}\left[\frac{l(l+1)}{2}r^2(\mathbf{v}.(\mathbf{H}_M \times \mathbf{r}) + (\mathbf{r}\times\omega)\cdot(\mathbf{H}_M \times \mathbf{r}))\right]. \tag{34}$$

$$S_l = \frac{-Gm^2 k_l \Delta t R_E^{2l+1} l(l+1)}{2\mu}\left[\frac{n}{a^{2l+3}}(1-e^2)^{1/2}\left(\frac{a}{r}\right)^{2l+5} - \frac{\omega\cos I}{a^{2l+3}}\left(\frac{a}{r}\right)^{2l+3}\right]. \tag{35}$$

Finally, the expression for $W_l$ is given by:

$$W_l = \frac{-Gm^2 k_l \Delta t R_E^{2l+1}}{\mu r^{2(l+3)}}\frac{1}{mna^2(1-e^2)^{1/2}}\left[\frac{l(l+1)}{2}r^2\mathbf{H}_M\cdot(\mathbf{r}\times\omega)\right], \tag{36}$$

$$W_l = \frac{-Gm^2 k_l \Delta t R_E^{2l+1} l(l+1)\omega}{2\mu a^{2l+3}}\left(\frac{a}{r}\right)^{2l+3}\cos(\bar{\omega}+\nu)\sin I \tag{37}$$

The dynamic equations, according to the Gaussian formalism, are:

$$\frac{da}{dt} = \frac{2}{n(1-e^2)^{1/2}}\left(Re\sin\nu + (1-e^2)\frac{a}{r}S\right), \tag{38}$$

$$\frac{de}{dt} = \frac{(1-e^2)^{1/2}}{na}\left[R\sin\nu + S\left(\frac{1}{e} - \frac{r}{ea} + \cos\nu\right)\right], \tag{39}$$

and

$$\frac{di}{dt} = \frac{r\cos(\bar{\omega}+\nu)}{na^2(1-e^2)^{1/2}}W, \tag{40}$$

where $i$ corresponds to the inclination of the Moon's orbital plane with respect to the inertial plane which is orthogonal to the total angular momentum of the Earth-Moon system.

We stress the fact that we are considering no external force with respect to this system.

Now, following the formalism of the coefficients of Hansen, we expand Eqs. (38), (39), and (40) as a finite series of powers of $e$. For this purpose, consider the following time averages:

$$\langle R_l e\sin\nu\rangle = \frac{-Gm^2 k_l \Delta t n(l+1)^2 e^2 R_E^{2l+1}}{\mu a^{2l+3}(1-e^2)^{1/2}}\left\langle\left(\frac{a}{r}\right)^{2(l+2)}\sin^2\nu\right\rangle = \frac{-Gm^2 k_l \Delta t n(l+1)^2 e^2 R_E^{2l+1}}{2\mu a^{2l+3}(1-e^2)^{1/2}}\left(\left\langle\left(\frac{a}{r}\right)^{2l+4}\right\rangle - \left\langle\left(\frac{a}{r}\right)^{2l+4}\cos(2\nu)\right\rangle\right), \tag{41}$$

$$\left\langle\left(\frac{a}{r}\right)^j\right\rangle = H(j,0), \tag{42}$$

$$\left\langle\left(\frac{a}{r}\right)^j\cos\nu\right\rangle = H(j,1), \tag{43}$$

$$\left\langle\left(\frac{a}{r}\right)^j\cos(2\nu)\right\rangle = H(j,2), \tag{44}$$

where the following relation is valid:

$$H(j,2) = H(j,0) - \frac{2(1-e^2)}{e(j-1)}H(j+1,1). \tag{45}$$

So, we can write the relations given below:

$$H(j,0) = \frac{A_j}{(1-e^2)^{j-3/2}}, \tag{46}$$

$$H(j,1) = \frac{B_j}{(1-e^2)^{j-3/2}}. \tag{47}$$

The coefficients $A_j$ and $B_j$, for some values of $j$, are given by the next tables:

| $j$ | $A_j$ |
|---|---|
| 6 | $1+3e^2+\frac{3}{8}e^4$ |
| 7 | $1+5e^2+\frac{15}{8}e^4$ |
| 8 | $1+\frac{15}{2}e^2+\frac{45}{8}e^4+\frac{5}{16}e^6$ |
| 9 | $1+\frac{21}{2}e^2+\frac{105}{8}e^4+\frac{35}{16}e^6$ |
| 10 | $1+14e^2+\frac{210}{8}e^4+\frac{35}{4}e^6+\frac{35}{128}e^8$ |

(48)

| $j$ | $Bj$ |
|---|---|
| 6 | $2e + \frac{3}{2}e^3$ |
| 7 | $\frac{5}{2}e + \frac{15}{4}e^3 + \frac{5}{16}e^5$ |
| 8 | $3e + \frac{30}{4}e^3 + \frac{15}{8}e^5$ |
| 9 | $\frac{7}{2}e + \frac{105}{8}e^3 + \frac{105}{16}e^5 + \frac{35}{128}e^7$ |
| 10 | $4e + 21e^3 + \frac{35}{2}e^5 + \frac{35}{16}e^7$ |

(49)

Hence:

$$\left\langle R_l e \sin v \right\rangle = \frac{-Gm^2 k_l \Delta t \; n(l+1)^2 e^2 R_e^{2l+1}}{2\mu a^{2l+3}(1-e^2)^{1/2}} \left(H(2l+4,0) - H(2l+4,2)\right), \tag{50}$$

that is,

$$\left\langle R_l e \sin v \right\rangle = \frac{-Gm^2 k_l R_E^{2l+1} \Delta t (l+1)^2 (1-e^2)^{1/2} en}{\mu(2l+3)a^{2l+3}} H(2l+5,1). \tag{51}$$

Analogously:

$$\left\langle \frac{a}{r} S_l \right\rangle = \frac{-Gm^2 k_l \Delta t R_E^{2l+1} l(l+1)n}{2\mu a^{2l+3}} \left[(1-e^2)^{1/2} H(2l+6,0) - \right.$$
$$\left. - \frac{\omega \cos I}{n} H(2l+4,0)\right]. \tag{52}$$

The most important term in the disturbing potential is the term of the second degree (Love, 1944, pp. 261). So, for $l = 2$:

$$\left\langle R_2 e \sin v \right\rangle = \frac{-9Gm^2 k_2 R_E^5 \Delta t n e^2}{2\mu a^7 (1-e^2)^7} \left[1 + \frac{15}{4}e^2 + \frac{15}{8}e^4 + \frac{5}{64}e^6\right], \tag{53}$$

and

$$\left\langle \frac{a}{r} S_2 \right\rangle = \frac{-3Gm^2 k_2 R_E^5 \Delta t n}{\mu a^7} \left[\frac{1}{(1-e^2)^8}\left(1 + 14e^2 + \frac{105}{4}e^4 + \right.\right.$$
$$\left.\left. + \frac{35}{4}e^6 + \frac{35}{128}e^8\right) - \frac{\omega}{n}\frac{\cos I}{(1-e^2)^{13/2}}\left(1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6\right)\right]. \tag{54}$$

Hence, considering

$$X = \frac{a}{R_E}, \tag{55}$$

and

$$P = 2\pi \left(\frac{R_E^3}{GM}\right)^{1/2} \tag{56}$$

we have

$$\frac{dX}{dt} = \frac{24\pi^2 k_2 m^2 \Delta t}{M\mu P^2 X^7}\left[\frac{-1}{(1-e^2)^{15/2}}\left(1 + \frac{31}{2}e^2 + \frac{255}{8}e^4 + \right.\right.$$
$$\left.\left. + \frac{185}{16}e^6 + \frac{25}{64}e^8\right) + \frac{\omega \cos I}{n(1-e^2)^6}\left(1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6\right)\right]. \tag{57}$$

To obtain the second dynamic equation that gives the time evolution of the eccentricity, we need to expand the terms $R_l \sin v$ and $S_l(e^{-1} - r/(ea) + \cos v)$ as follows:

$$\left\langle R_l \sin v \right\rangle = \frac{-Gm^2 k_l R_E^{2l+1} \Delta t (l+1)^2 (1-e^2)^{1/2} n}{\mu(2l+3)a^{2l+3}} H(2l+5,1). \tag{58}$$

The second expansion is:

$$\left\langle S_l \left(\frac{1}{e} - \frac{r}{ea} + \cos v\right)\right\rangle =$$
$$= \frac{-Gm^2 k_l \Delta t R_E^{2l+1} l(l+1)n}{2e\mu a^{2l+3}}\left[(1-e^2)^{1/2}\left(H(2l+5,0) - H(2l+4,0) + \right.\right.$$
$$\left.\left. + eH(2l+5,1)\right) - \frac{\omega}{n}\cos I \left(H(2l+3,0) - H(2l+2,0) + eH(2l+3,1)\right)\right]. \tag{59}$$

By using the following recurrence formula

$$H(j,0) = \frac{1}{1-e^2}\left[H(j-1,0) + eH(j-1,1)\right], \tag{60}$$

we have, analogously, for $l = 2$:

$$\frac{de}{dt} = 12\pi^2 k_2 \frac{m^2 \Delta t}{M\mu P^2 X^8}\left[\frac{-1}{(1-e^2)^{13/2}}\left(9e + \frac{135}{4}e^3 + \frac{135}{8}e^5 + \right.\right.$$
$$\left.\left. + \frac{45}{64}e^7\right) + \frac{\omega}{n}\frac{\cos I}{(1-e^2)^5}\left(\frac{11}{2}e + \frac{33}{4}e^3 + \frac{11}{16}e^5\right)\right]. \tag{61}$$

In order to obtain the third and last differential equation that describes the dynamical system in question we need to observe that

$$\sin I = \frac{H}{H_E}\sin i, \tag{62}$$

that is

$$\sin I = \frac{T\,(GMm^2 R_E)^{1/2}\ \sin i}{C\omega} \tag{63}$$

where $C$ is the main inertial moment of the Earth, or inertial polar moment, which is given by $C = H_E/\omega$. So,

$$\frac{di}{dt} = \frac{-6\pi^2\, k_2 m^2\, \Delta t T\ \sin i}{M^2\, P^2\, \alpha\, X^{13/2}\, (1-e^2)^5}\left(1 + 3e^2 + \frac{3}{8}e^4\right), \tag{64}$$

where

$$T = H\,(G\,M\,m^2 R_E)^{1/2}, \tag{65}$$

and

$$\alpha = \frac{C}{M\,R_E^2} \tag{66}$$

## THE SEMIEMPIRICAL MODEL

Consider the following equation:

$$\mathbf{H} = \mathbf{H}_M + \mathbf{H}_E \tag{67}$$

where $\mathbf{H}$ is the total moment of the Earth-Moon system, and $\mathbf{H}_M$ and $\mathbf{H}_E$ are, respectively, the moment of the Moon and the moment of the Earth. If we perform, in Eq. (67), a dot product by $\mathbf{H}_M$ we have the following relation:

$$H \cos i - H_E \cos I = H_M, \tag{68}$$

which is equivalent to

$$\frac{\omega}{n} \cos I = \frac{1}{\beta}\,[T\,X^{3/2}\ \cos i - X^2\ (1-e^2)^{1/2}], \tag{69}$$

or

$$\omega = \frac{1}{\beta \cos I}\left(\frac{GM}{R_E^3}\right)^{1/2}\ [T\ \cos i - X^{1/2}\ (1-e^2)^{1/2}], \tag{70}$$

where

$$\beta = \alpha\,\frac{M}{m}. \tag{71}$$

Thus:

$$\cos I = \frac{T\ \cos i - X^{1/2}\ (1-e^2)^{1/2}}{(T^2 - 2T\,X^{1/2}\,(1-e^2)^{1/2}\ \cos i + X\,(1-e^2))^{1/2}}. \tag{72}$$

Deriving Eq. (70) with respect to t (time), we have the following:

$$\dot\omega = \frac{m\,(GM)^{1/2}}{M\,\alpha\,R_E^{3/2}\,\cos I}\left[-T\,\dot i\,\sin i - \frac{\dot X\,(1-e^2)^{1/2}}{2X^{1/2}} + \frac{e\dot e X^{1/2}}{(1-e^2)^{1/2}} - \right.$$
$$\left. - \frac{T\ \cos i}{\cos I}\,\frac{d}{dt}(\cos I) + \frac{X^{1/2}\ (1-e^2)^{1/2}}{\cos I}\,\frac{d}{dt}(\cos I)\right], \tag{73}$$

where

$$\frac{d}{dt}(\cos I) = \frac{1}{2}\left(\frac{-T^2\,\dot X\,(1-e^2)^{1/2}\ \sin^2 i}{X^{1/2}} + \frac{2T^2\,X^{1/2}\,e\dot e\ \sin^2 i}{(1-e^2)^{1/2}} + \right.$$
$$\left. + 2\dot i T^2\ \sin i\,(X^{1/2}\,(1-e^2)^{1/2}\ \cos i - T)\right)(T^2 - 2T\,X^{1/2}\,(1-e^2)^{1/2}\ \cos i +$$
$$+ X\,(1-e^2))^{3/2}. \tag{74}$$

John W. Wells (1963) was the first one to observe that paleontology allows to determine the number of days in the year for different geological periods, from Cenozoic to Cambrian. Some corals that lived hundreds of millions of years ago had annual growth increments on their upper surface, ascribed to seasonal temperature changes. Assuming that the period of the Earth's revolution around the Sun has been constant, and that its period of rotation on its polar axis is changing by tidal forces, it is possible to estimate the number of days per year at the time when those corals lived. Hence, it is possible to estimate that the length of the day has been increasing throughout geological time and that the number of days in the year has been decreasing. Wells calculates, for example, that the number of days of the Cambrian year (600 × $10^6$ years ago) was of the order of 420.

In Fig. 1 it follows the graphic which appears in Wells' paper (op. cit.).

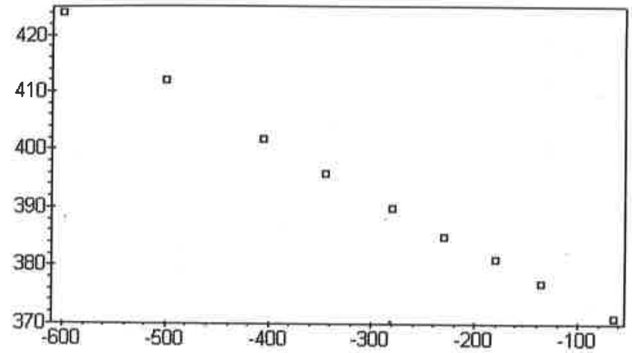


**Figure 1** - This graphic corresponds to the data obtained from paleonthological observations, according to Wells. The horizontal axis is the geological time (millions of years). The vertical axis is the number of days in a year.

***Figura 1*** - *Este gráfico corresponde aos dados obtidos de observações paleontológicas, de acordo com Wells. O eixo horizontal representa o tempo geológico em milhões de anos. O eixo vertical é o número de dias em um ano.*

Based on Wells' paper, we note a linear dependence between $\omega$ and $\dot{\omega}$, where $\omega$ is the angular velocity of the Earth's rotation. In Mignard's thesis such a relation is also presented. Hence:

$$\dot{\omega} = K\,\omega \tag{75}$$

where $K$=-0.00019/($10^6$ years).

If we compare Eq. (75) with Eq. (73), which gives the expression for $\dot{\omega}$, we can state an expression for the tidal time delay $\Delta t$ as a function of $X$, $e$, $i$ and $\omega$ as follows:

$$\Delta t = \frac{K\,\omega}{F\,(X,\,e,\,i)}, \tag{76}$$

where:

$$F\,(X,e,i) = \frac{m(GM)^{1/2}}{M\,\alpha\,R_E^{3/2}\,\cos I}\left[-T\,\dot{i}\,\sin i - \frac{\dot{X}\,(1-e^2)^{1/2}}{2X^{1/2}} + \frac{e\dot{e}X^{1/2}}{(1-e^2)^{1/2}} - \right.$$
$$\left. - \frac{T\cos i}{\cos I}\,\overline{D} + \frac{X^{1/2}\,(1-e^2)^{1/2}}{\cos I}\,\overline{D}\right], \tag{77}$$

where:

$$\overline{X} = \frac{24\pi^2 k_2\,m^2}{M\,\mu\,P^2 X^7}\left[\frac{-1}{(1-e^2)^{15/2}}\left(1 + \frac{31}{2}e^2 + \frac{255}{8}e^4 + \right.\right.$$
$$\left.\left. + \frac{185}{16}e^6 + \frac{25}{64}e^8\right) + \frac{\omega\cos I}{n\,(1-e^2)^6}\left(1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6\right)\right], \tag{78}$$

$$\overline{e} = 12\pi^2 k_2\,\frac{m^2}{M\,\mu\,P^2\,X^8}\left[\frac{-1}{(1-e^2)^{13/2}}\left(9e + \frac{135}{4}e^3 + \frac{135}{8}e^5 + \right.\right.$$
$$\left.\left. + \frac{45}{64}e^7\right) + \frac{\omega}{n}\,\frac{\cos I}{(1-e^2)^5}\left(\frac{11}{2}e + \frac{33}{4}e^3 + \frac{11}{16}e^5\right)\right], \tag{79}$$

$$\dot{i} = \frac{-6\pi^2 k_2\,m^2\,T\sin i}{M^2\,P^2\,\alpha\,X^{13/2}\,(1-e^2)^5}\left(1 + 3e^2 + \frac{3}{8}e^4\right), \tag{80}$$

and

$$\overline{D} = \frac{1}{2}\left(\frac{-T^2\,\overline{X}\,(1-e^2)^{1/2}\,\sin^2 i}{X^{1/2}} + \frac{2T^2\,X^{1/2}\,e\overline{e}\sin^2 i}{(1-e^2)^{1/2}} + \right.$$
$$\left. + 2\,\dot{i}\,T^2 \sin i\,(X^{1/2}\,(1-e^2)^{1/2}\cos i - T)\right)\left(T^2 - 2T\,X^{1/2}(1-e^2)^{1/2}\cos i + \right.$$
$$\left. + X\,(1-e^2)\right)^{-3/2}. \tag{81}$$

Note that Eq. (76) is only a first attempt to give an expression for $\Delta t$. If we consider $\Delta t$ as a function defined on a time interval, then Eqs. (41), (52), (58), and (59) should be rewritten, as well as all the remaining equations derived from them.

Due to the high non-linearity of the dynamical system that we are analysing, the numerical integration is very difficult to be performed in a personal computer, if we are interested in an integration that gives information about a remote past like one billion years ago. One manner to facilitate the numerical integration is to rewrite the dynamic equations (57), (61), and (64) as

$$\frac{dX}{di} = \frac{-4M\alpha}{\mu T\,X^{1/2}\sin i}\left[\frac{-1}{(1-e^2)^{5/2}}\left(1 + \frac{31}{2}e^2 + \frac{255}{8}e^4 + \frac{185}{16}e^6 + \right.\right.$$
$$\left.\left. + \frac{25}{64}e^8\right) + \frac{\omega\cos I}{n(1-e^2)}\left(1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6\right)\right]\left(1 + 3e^2 + \frac{3}{8}e^4\right)^{-1}, \tag{82}$$

$$\frac{de}{di} = \frac{-2M\alpha}{\mu T X^{3/2} \sin i}\left[\frac{-1}{(1-e^2)^{3/2}}\left(9e + \frac{135}{4}e^3 + \frac{135}{8}e^5 + \frac{45}{64}e^7\right) + \right.$$

$$\left. + \frac{\omega \cos I}{n}\left(\frac{11}{2}e + \frac{33}{4}e^3 + \frac{11}{16}e^5\right)\right]\left(1 + 3e^2 + \frac{3}{8}e^4\right)^{-1}, \qquad (83)$$

and

$$\frac{dt}{di} = \frac{-M^2 P^2 \alpha X^{-13/2}(1-e^2)^5}{6\pi^2 k_2 m^2 \Delta t T \sin i}\left(1 + 3e^2 + \frac{3}{8}e^4\right)^{-1}, \qquad (84)$$

where the expression

$$\frac{\omega \cos I}{n}$$

is given by Eq. (69).

It is interesting to observe from Eqs. (82), (83), and (84) that the triples $(X,e,i)$ do not depend on the values of $\Delta t$. The time delay has an influence only on the time scale of our problem.

To perform the numerical integration we consider the following data: $\pi = 3.14159$, $k_2 = 0.28$, $m = 7.36\times10^{22}$ $kg$, $M = 5.98\times10^{24}$ $kg$, $\mu = 7.27\times10^{22}$ $kg$, $n_G = 3.91\times10^{10}$ $(10^6\ \text{years})^{-1}$, $G = 6.64\times10^{16}$ $m^3/((10^6\ \text{years})^2\ kg)$, $P = 1.60\times10^{-10}\times10^6$ *years*, $R_E = 6.371\times10^6$ $m$, $\alpha = 0.400$, and $C = 8.043\times10^{37}$ $kg\ m^2$.

We use the fourth order Runge-Kutta method to solve our system of differential equations given by Eqs. (82), (83), and (84). We integrate Eqs. (82), and (83) in order to obtain the triples $(X,e,i)$. Then we substitute these values in Eq. (84) to obtain time scales. To facilitate the calculations we assume as time unit $10^6$ years.

## DISCUSSIONS AND CONCLUSIONS

If we consider $\Delta t$ as a constant with respect to time $t$, as Mignard does (1979; 1980), and assume the initial conditions of the present eccentricity $e = 0.05$ of the Moon's orbit, distance $X = 60.25$ from the Earth to the Moon, and inclination $i = 0.0695$, we have as a point of 'maximum approximation' $t = -2.042$ billion years, performing an integration with positive step for $i$ equal to $10^{-6}$ (that is, an integration to the past). The meaning of 'maximum approximation' can be understood if we see the graphics given in Figs. 2, 4, and 6. If we consider

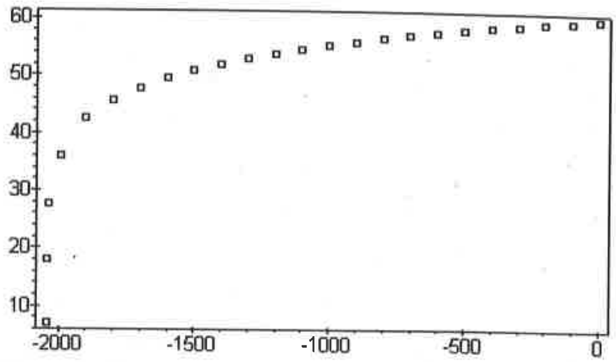


**Figure 2** - This graphic corresponds to the dynamics of the Earth-Moon system with a constant tidal time delay. Only some points of the numerical integration were plotted. The horizontal axis is the geological time (millions of years). The vertical axis is the distance between the Earth and the Moon (the quotient between the semi-major axis of the Moon's orbit and the Earth's equatorial radius).

***Figura 2** - Este gráfico corresponde à dinâmica do sistema Terra-Lua com um atraso de resposta constante. Apenas alguns pontos da integração numérica foram plotados. O eixo horizontal é o tempo geológico em milhões de anos. O eixo vertical é a distância entre a Terra e a Lua (a razão entre o semi-eixo maior da órbita da Lua e o raio equatorial da Terra).*

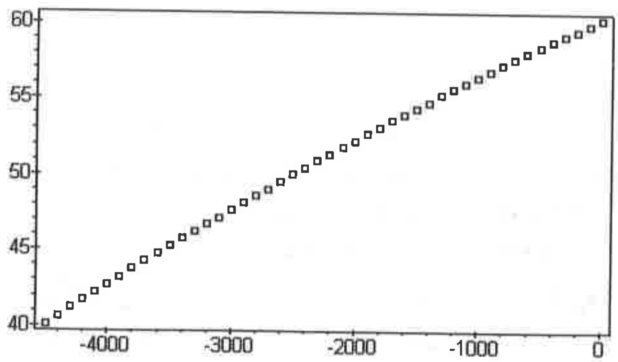


**Figure 3** - This graphic corresponds to the dynamics of the Earth-Moon system with a varying tidal time delay. Only some points of the numerical integration were plotted. The horizontal axis is the geological time (millions of years). The vertical axis is the distance between the Earth and the Moon (the quotient between the semi-major axis of the Moon's orbit and the Earth's equatorial radius).

***Figura 3** - Este gráfico corresponde à dinâmica do sistema Terra-Lua com um atraso de resposta variável. Apenas alguns pontos da integração numérica foram plotados. O eixo horizontal é o tempo geológico em milhões de anos. O eixo vertical é a distância entre a Terra e a Lua (a razão entre o semi-eixo maior da órbita da Lua e o raio equatorial da Terra).*

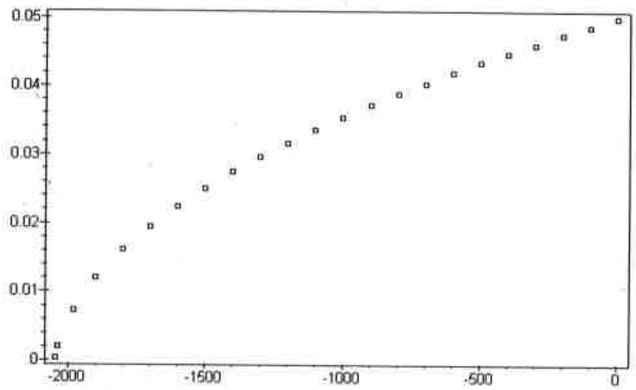


**Figure 4** - Constant tidal time delay. The horizontal axis is the geological time (millions of years). The vertical axis is the eccentricity of the Moon's orbit.

***Figura 4** - Atraso de resposta constante. O eixo horizontal é o tempo geológico em milhões de anos. O eixo vertical é a excentricidade da órbita da Lua.*

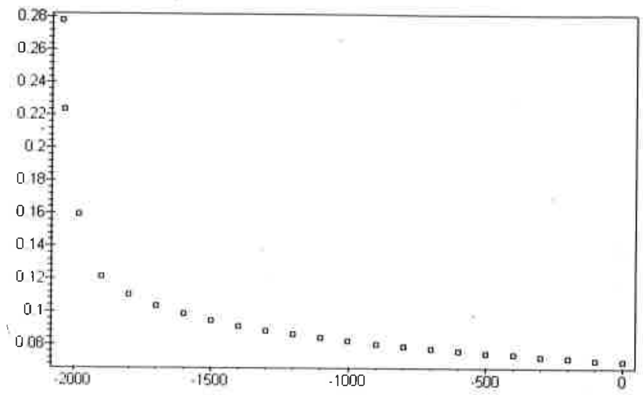


**Figure 6** - Constant tidal time delay. The horizontal axis is the geological time (millions of years). The vertical axis is the inclination $i$ of the Moon's orbit.

***Figura 6** - Atraso de resposta constante. O eixo horizontal é o tempo geológico em milhões de anos. O eixo vertical é a inclinação* i *da orbita da Lua.*

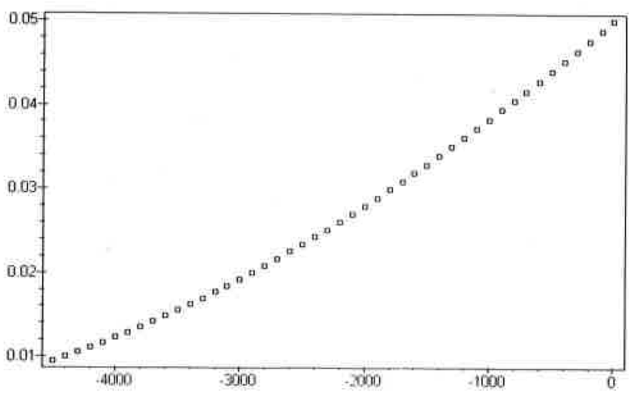


**Figure 5** - Varying tidal time delay. The horizontal axis is the geological time (millions of years). The vertical axis is the eccentricity of the Moon's orbit.

***Figura 5** - Atraso de resposta variável. O eixo horizontal é o tempo geológico em milhões de anos. O eixo vertical é a excentricidade da órbita da Lua*

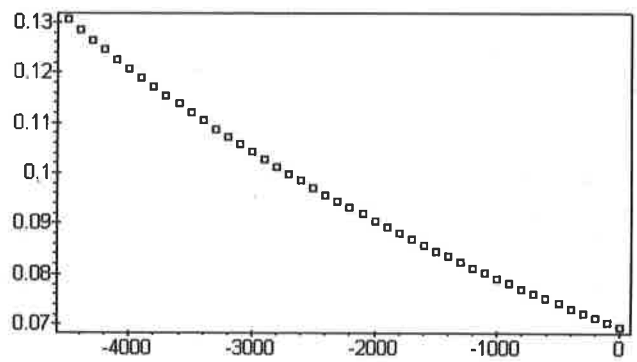


**Figure 7** - Varying tidal time delay. The horizontal axis is the geological time (millions of years). The vertical axis is the inclination $i$ of the Moon's orbit.

***Figura 7** - Atraso de resposta variável. O eixo horizontal é o tempo geológico em milhões de anos. O eixo vertical é a inclinação* i *da órbita da Lua.*

the tidal time delay $\Delta t$ constant, the Earth-Moon system could not be more than 2.042 billion years old. Nevertheless, it is well known that the Moon formed 4.52 to 4.50 billion years ago (Lee et al., 1997). The reader could question the validity of our choice for the step of integration $10^{-6}$. Notwithstanding, we have performed the same numerical integration for other steps $h$, where $h << 10^{-6}$, with no significant changes in our final results. Hence, our choice for the value of the step of integration sounds very reasonable.

On the other hand, if we perform the numerical integration by considering an expression for $\Delta t$, given by Eq. (76), the resulting graphics are those presented in Figs. 3, 5, and 7, which show that a numerical integration to the distant past of 4.5 billion years is possible. This occurs because the time delay decreases to the past and increases to the future. Our calculations with a varying tidal time delay are more realistic than Mignard's model, if we compare them with the modern theories about the origin of the Earth-Moon system (Lee et al, op. cit.; Benz et al, 1986, 1987; Patterson, 1987). In other words, our semi-empirical model derived from Eq. (75) says that $X$ and $e$ decrease and $i$ increases to the distant past, but not so fast as it occurs in Mignard's model. The reason for that is the following: Eq. (75) provides us with *all* perturbations in the Earth-Moon system. Mignard's model accounts only for the perturbations caused by the Earth's tides, ignoring other effects like those caused by the Sun.

Our model does not allow to make numerical integration to the distant future, since $\Delta t$ increases to the future and, hence, Eq. (10) must be reviewed. Such an equation ignores the terms of superior order with respect to $\Delta t$ in the Taylor series that describes the perturbative potential, because the time delay is considered sufficiently small for that purpose. Nevertheless the numerical integration to the distant future is possible if we consider $\Delta t$ constant.

The reader could ask if a second order series for eccentricity would be enough. Yes, it would be. But we discovered that only after the numerical integration, when we noticed that eccentricity decreases to the past, according to the graphics in Fig. 4 and Fig. 5. So. Eqs. (82), (83), and (84) can be rewritten as:

$$\frac{dX}{di} = \frac{-4M\alpha}{\mu T X^{1/2} \sin i\,(1+3e^2)} \left[ \frac{(-1-\frac{31}{2}e^2)}{(1-e^2)^{5/2}} + \frac{\omega \cos I\,(1+\frac{15}{2}e^2)}{n(1-e^2)} \right], \tag{85}$$

$$\frac{de}{di} = \frac{-2M\alpha}{\mu T X^{3/2} \sin i} \left[ \frac{-9e}{(1-e^2)^{3/2}} + \frac{11e\,\omega \cos I}{2n} \right] (1+3e^2)^{-1}, \tag{86}$$

and

$$\frac{dt}{di} = \frac{-M^2 P^2 \alpha X^{-13/2}\,(1-e^2)^5}{6\pi^2 k_2 m^2 \Delta t\, T \sin i} (1+3e^2)^{-1}. \tag{87}$$

Another interesting point is an open problem related to a possible extension of our results to a system of three bodies such as the Earth-Moon-Sun system, although the contribution of the Moon to the tidal perturbation is considerably greater than the one due to the Sun.

As a final remark, we would like to point out that we may apply our model to other planet-satellite systems if we get the time delay and the Love numbers of those systems.


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## SOBRE O ATRASO DE RESPOSTA DE MARÉS DA TERRA

Desde antigas observações de eclipses, astrônomos têm observado que existe uma aceleração secular da Lua. Físicos e matemáticos como Laplace, Halley, Adams e mesmo o filósofo Kant, estudaram a respeito de possíveis modelos que explicassem tal perturbação na órbita lunar. Em nosso século, autores como Goldreich, Kaula, MacDonald e Mignard têm estudado sobre os efeitos de marés oceânicas e terrestres sobre os movimentos de rotação da Terra e de translação da Lua. Usualmente, o efeito perturbativo das marés sobre a dinâmica de sistemas planeta-satélite é calculado utilizando-se o formalismo dos números de Love. No caso do formalismo de Darwin, um atraso de resposta $\Delta t$ também é levado em conta para o sistema Terra-Lua. Tal atraso corresponde ao intervalo de tempo necessário para a maré se formar na Terra a partir do momento em que ocorre a perturbação em nosso planeta provocada pela Lua. O astrônomo francês Mignard desenvolveu um modelo para esta dinâmica que representa um misto do formalismo de Darwin com o dos números de Love. Em contrapartida, em 1963 Wells publicava um artigo no qual ele defendia a idéia de que o exame de fósseis possibilita a determinação de ciclos vitais de corais que viveram há milhões de anos. Uma vez que estes ciclos vitais são regulados por períodos astronômicos como o ano tropical, o mês sinódico e o dia solar, fica possível também determinar a duração destes intervalos de tempo na época em que estes corais viveram. No presente artigo apresentamos uma expressão para o atraso de resposta das marés da Terra, a qual leva em consideração as observações paleontológicas de Wells. Um modelo semi-empírico para a dinâmica do sistema Terra-Lua é obtido, considerando-se que a dissipação de energia e o atraso de resposta não são constantes em relação ao tempo. Em um trabalho anterior nosso estabelecemos um modelo similar considerando apenas a dinâmica do raio de uma órbita circular da Lua. Porém, no presente artigo três equações diferenciais descrevem a dinâmica de três variáveis, a saber, o raio médio da órbita lunar, a excentricidade da órbita e a inclinação do plano orbital da Lua em relação ao plano inercial ortogonal ao momento angular total do sistema Terra-Lua. Neste sentido, consideramos o sistema Terra-Lua isolado. Mignard já havia considerado que um atraso de resposta variável deveria ser mais realista que a hipótese de um $\Delta t$ constante. Em um estudo apenas heurístico, Mignard concluiu que o atraso de resposta deve ter sido menor no passado distante de milhões de anos atrás. Verificamos, no processo de integração numérica de nossas equações diferenciais, que Mignard estava certo. Comparamos nossa descrição com aquela que considera o atraso de resposta de maré constante em relação ao tempo e mostramos que nosso modelo é mais consistente com as modernas teorias sobre a formação do sistema Terra-Lua. Isso porque no modelo de Mignard, o sistema Terra-Lua não poderia ter mais do que 2,042 bilhões de anos, ao passo que nosso modelo semi-empírico é compatível com as atuais teorias segundo as quais a Lua surgiu há cerca de 4,50 bilhões de anos. No final do trabalho propomos algumas questões em aberto que devem ser respondidas em trabalhos futuros, tais como a extensão deste modelo para sistemas de três corpos e a aplicação de nossas idéias em outros sistemas planeta-satélite. Outra questão em aberto refere-se à expressão para o atraso de resposta. Nosso modelo pressupõe que $\Delta t$ deve ser suficientemente pequeno de modo a podermos desprezar parcelas de ordem igual ou superior a $\mathcal{O}(\Delta t^2)$ na expressão do potencial perturbador. Isso significa que nosso modelo de atraso de resposta variável não permite integrações numéricas para o futuro distante, pois $\Delta t$ aumenta consideravelmente no futuro. No entanto, sugerimos de que forma podemos melhorar nossa teoria de modo a permitir integrações para o futuro distante.

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