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LAYER-INDUCED ELASTIC ANISOTROPY - PART 1: FORWARD RELATIONS BETWEEN CONSTITUENT PARAMETERS AND COMPOUND MEDIUM PARAMETERS*

Klaus Helbig

An important reason for elastic anisotropy of geological media is layering on a scale that is small to the wavelengths. If the parameters of the layering are known, the "effective parameters" of the compound medium can be derived from the parameters of the constituents. If the layering is periodic with the period small compared to the wavelengths, the compound medium can be replaced by a homogeneous medium. One way to achieve this description is to re-formulate Hooke's law. First, the six components of stress and strain that vary from constituent to constituent are expressed in terms of the six components that are, in view of the continuity condition, constant throughout the layer. Second, the effective stress- and strain components are determined as the thickness-weighted arithmetic averages of the corresponding constituents' components. Third, the standard form of Hooke's law (stress as a function of strain) is re-established. This algorithm can be applied to constituents of any type. For seismic applications, the most important layering is that of isotropic layers. There are two significant situations that afford a simplification: (i) if all constituents have the same shear stiffness, the replacement medium is homogeneous; (ii) if all constituents have the same Poisson's ratio, wave propagation in a relatively wide cone about the axis of symmetry cannot be distinguished from wave propagation in an isotropic medium.

Key words: Elastic anisotropy; Exploration seismics; Wave propagation.

ANISOTROPIA ELÁSTICA INDUZIDA PELA ESTRATIFICAÇÃO - PARTE 1: RELAÇÕES DIRETAS ENTRE PARÂMETROS DOS CONSTITUINTES E DO MATERIAL COMPOSTO - *Uma importante causa da anisotropia elástica em materiais geológicos é a estratificação em camadas menores do que o comprimento de onda. Quando os parâmetros da estratificação são conhecidos, os "parâmetros efetivos" do material composto pode ser derivado dos parâmetros dos elementos constituintes. No caso da estratificação ser periódica, com periodos menores do que o comprimento de onda, o material composto pode ser substituido por um meio homogêneo. A maneira para se obter estas condições é através da reformulação da Lei de Hooke. Primeiramente, os seis componentes de esforço e deformação, que variam entre os elementos constituintes, são expressos em termos dos seis componentes, que são constantes em toda a camada, em virtude da condição de continuidade. Em seguida, o componentes efetivos do esforço e deformação são determinados através de média aritmética ponderada pela espessura de cada componente dos elementos constituintes. Por fim, a forma usual da Lei de Hooke (esforço em função da deformação) é restabelecida. Este procedimento pode ser aplicado a elementos constituintes de qualquer tipo. Em aplicações sísmicas, as camadas isotrópicas são as mais importantes estratificações encontradas. Existem duas importantes situações que permitem uma simplificação do problema: (i) quando todos os elementos constituintes tiverem a mesma rigidez de cizalhamento, pode-se utilizar um meio homogêneo; (ii) quando todos os elementos constituintes tiverem a mesma razão de Poisson, a propagação de onda em um relativamente amplo cone ao redor do eixo de simetria não é distinguível da propagação de onda em um meio isotrópico.*

***Palavras-chave**: Anisotropia elástica; Sísmica de exploração; Propagação de ondas.*

Kiebitzrain 84
D 30657 Hannover
Germany
helbig@real-net.de

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INTRODUCTION

During an in-mine survey in 1953-54 I came across the observation that the velocity of P-waves in Devonian schist could vary by more than 20 % with direction, and that there were S-waves with different velocities. This observation initiated my interest in seismic anisotropy. Two years later I completed a thesis on wave propagation in anisotropic media, which was concerned primarily with transverse anisotropy, and more specifically with the anisotropy due to layering. A major finding (reported earlier by Bruggeman, 1937; Riznichenko, 1948, 1949; and Postma, 1955) was that such layering nearly always leads to significant anisotropy.

My mentor, Theodor Krey, took the draft on a brief trip to Rio de Janeiro and read it on the plane. He came back with - besides the approval - one significant insight concerning layer-induced anisotropy (Krey & Helbig, 1956). I shall refer to it under the name of *K-medium.*

Anisotropy is significant for exploration seismics for two reasons:

- Many of the standard algorithms of the processing suite (e.g., NMO-correction, stacking and velocity determination, DMO-correction, migration) implicitly assume a spherical wave front. If the wave fronts are not spherical, these algorithms will not work as expected, and one has at least to apply corrections.
- If anisotropy is observed, one can try to determine the cause of this anisotropy. In this way one would obtain information at a scale beyond the resolution of the seismic method.

THE CONCEPT OF THE "EFFECTIVE MEDIUM"

The cause of material anisotropy is *always* some internal structure on a scale that is small to the resolution of the method applied. This does even cover the anisotropy of crystals: the internal structure is the arrangement of the ions on a 3D-lattice on a scale that is small compared to optical wavelengths (though not small to the wavelengths of x-rays).

Internal structures in geological media that can lead to anisotropy are shown in the first panel of Fig. 1: orientated cracks, lamination (periodic sequences of thin layers), parallel fractures, oriented grains, and clay (orientation of plate-like minerals). To any observation that does not resolve the elements of the internal structures, these media appear to be transversely isotropic, i.e., homogeneous with an elastic matrix as shown in the second panel.

Since at the resolution of acquisition the heterogeneous, internally structured medium gives rise to the same observations as a homogeneous anisotropic medium, we can replace the heterogeneous medium in all further steps of data processing and interpretation with the *effective* replacement medium. Admittedly, anisotropic media are more difficult to deal with than isotropic ones, but the saving in complexity outweighs this disadvantage by far: the original heterogeneous medium is in many cases intractable.

One often calls the anisotropy of an effective medium "apparent anisotropy" or "quasi-anisotropy" to distinguish it from "intrinsic" (or genuine) anisotropy. This distinction is unnecessary, and even the term "intrinsic anisotropy" should be used only in a pragmatic sense. A standard example for genuine anisotropy are crystals, but they owe their anisotropy precisely to the ordered internal structure of the ions..

The term "intrinsic anisotropy" serves a purpose if it is used to indicate anisotropy due to structures that are several orders of magnitude below the resolution of the method. In exploration seismics it is used in this sense: anisotropy deduced from fluctuation of elastic parameters on the scale of about 30 cm (the resolution of the continuous velocity log) is seen in terms of elastic medium theory, anisotropy due to structures on an even smaller scale (e.g., the platelets in clay) is called "intrinsic". That this is not intended to disregard the internal structure as cause of the anisotropy of clays is obvious: the visual core inspection and the thin sections of the sedimentologists are as accessible to the seismic interpreter as are the continuous velocity logs.

In addition to material anisotropy, there are *conditions* where the propagation of waves is anisotropic even if the medium is isotropic. Such conditions could be moving coordinate systems such as the propagation of sound in air or water, either at rest observed from a moving platform, or in a moving medium (wind, current) observed from a fixed platform. The propagation of electromagnetic waves in plasmas becomes anisotropic under the influence of an external magnetic field. These types of anisotropy are outside the scope of this paper.

EFFECTIVE MEDIUM THEORY IN THE CONTEXT OF EXPLORATION SEISMICS

This article is concerned with the link between a sequence of homogeneous layers and the effective medium it can be replaced with. The deduction of the parameters of the original heterogeneous medium from those of the replacement medium will be discussed in a companion article prepared by the author. To place these two aspects into the context of exploration seismics, a brief discussion of the further steps is indicated in Fig. 1.

Due to the anisotropy expressed in the structure of the elastic matrix of the second panel, wave fronts are not spherical, and polarization vectors of P-and S-waves are not aligned with wave normals and wave surface, respectively (third panel). This leads to observable effects. A typical observation (one that can be explained only by anisotropy) is related to two distinct shear wave arrivals with different polarization (fourth panel).

The step from structure to elastic stiffness is done by a theory-based algorithm. Note that in the figure five common geological causes for seismic anisotropy are indicated. This list is by no means exhaustive. This article concentrates on one cause only, i.e., on fine layering, and on the algorithm to convert layer sequences into sets of stiffnesses. The reader should be aware of the non-uniqueness of this cause: one set of elastic stiffness might be generated by a variety of causes.

One purpose in observing anisotropy *effects* is to deduce from them (to invert them for) the *cause* of anisotropy, for instance, for the characterization of reservoirs. This involves several steps:

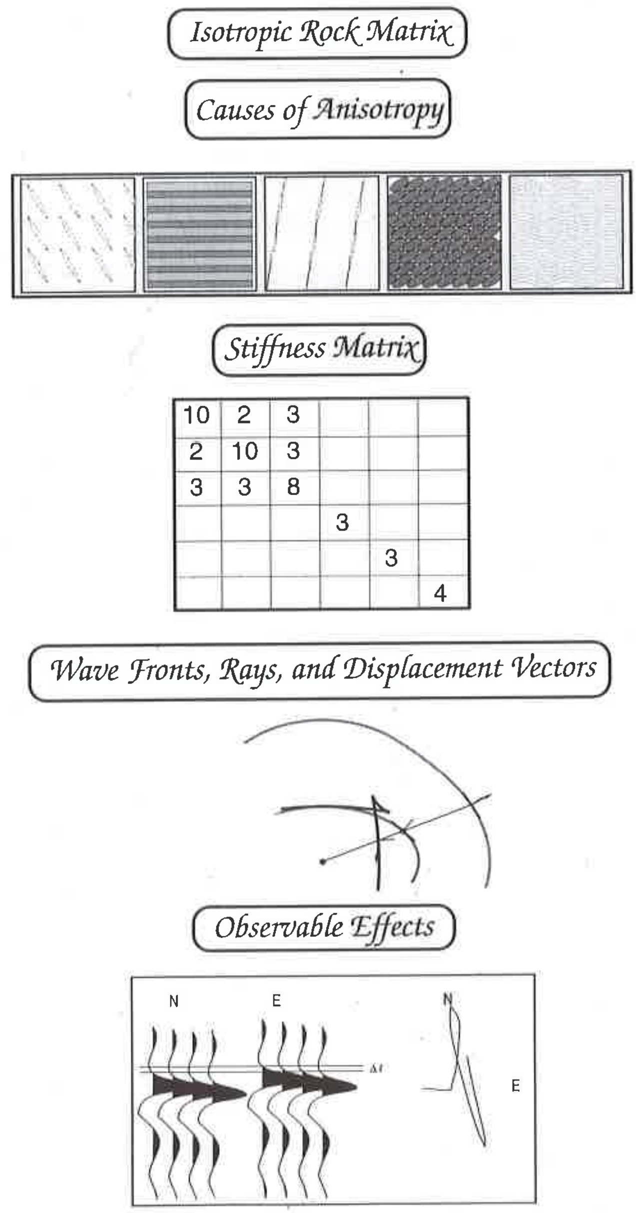

Figure 1 - *Panel 1(top):* There are many causes of anisotropy. The five causes indicated are, from left to right: oriented cracks, fine layering, parallel fractures, oriented sand grains, oriented clay minerals. / *Panel 2:* Each of the causes leads to a set of elastic stiffness that corresponds to "transverse isotropy". / *Panel 3:* Wave fronts are not spheres, rays are not perpendicular to wave fronts, displacement vectors form oblique angles, and there are two distinct shear waves. / *Panel 4:* Most observable effects are subtle, with the exception of the two distinct shear arrivals with orthogonal polarization.

Figura 1 - Painel 1 (topo): *Existem várias causas para anisotropia. Cinco são indicadas da esquerda para a direita do painel: rachaduras orientadas; acamamento fino; fraturas paralelas; grãos de areia orientados; minerais de argila orientados.* / Painel 2: *Cada uma das causas leva a um conjunto de rigidez elástica que corresponde a "isotropia transversa".* / Painel 3: *Frentes de onda não são esferas, raios não são perpendiculares às frentes de onda, vetores de deslocamentos formam ângulos oblíquos, e existem duas ondas de cizalhamento distintas.* / Painel 4: *A maior parte dos efeitos observáveis são sutis, com a exceção de duas chegadas de cizalhamento distintos, com polarização ortogonal.*

(i) to invert the effects, observed over a limited acquisition aperture, to the full slowness/ polarization information over a sufficiently large spatial range,
(ii) to invert the slowness/polarization information to the set of elastic stiffness, and
(iii) to invert the elastic stiffness to the causes.

Such an inversion is important, since it can reveal information on a scale below the resolution of the seismic method. In the companion article being prepared, the possibilities and limitations of the last step are discussed, again only for anisotropy due to layering. It is assumed that other steps can be done uniquely and with sufficient accuracy, and that all other causes of anisotropy can be excluded. Even under these restrictions the inversion is not unique. The underlying reason for the difficulty of inversion is the nature of the forward algorithm: it is expressed in terms of averages over certain combinations of elastic properties. To invert averages in terms of the individual data is generally not possible. That an inversion, though in a restricted sense, is possible at all is due to the combination of several averages.

This restriction of the inversion does not mean that the effort is in vain. However, to be meaningful, such an inversion has to be accompanied by outside information. A possibility would be the restriction of constituent layers to a few realistic lithologies, or at least to a limited range of elastic parameters for the constituents. The algorithm and its restricted inversion allow such limitations.

THE FORWARD PROBLEM: THE PARAMETERS OF THE COMPOUND MEDIUM IN TERMS OF PARAMETERS OF THE CONSTITUENTS

This problem was solved several time since it was first attacked by Bruggeman (1937). Postma (1955) treated only stacks of two isotropic constituents, while Bruggeman (1937), Backus (1962) and Helbig (1958) admitted unspecified numbers of transversely isotropic constituents with vertical axis and isotropic layers. Schoenberg & Muir (1989) gave an algorithm based on group theory that admitted *any* type and number of constituents. The algorithm proposed in this article solves the problem of Schoenberg & Muir (1989), but is derived without recourse to group theory.

The elastic stiffness of the compound medium can be derived from simple thought experiments. A few of these thought experiments are shown in Fig. 2, which shows also the idea underlying the concept of "effective media": the upper part of each of the four panels shows a representative block, a cube with a side equal to the period of the layer sequence, under different homogeneous stress-strain states. Since the cube is not resolved by the seismic method, we can talk only of the "compound" strain and stress (global ~, gross ~, overall ~). The details of the deformation are determined by application of HOOKE's law to the individual tensors of the constituents and by the rules of continuity, while the relationship between the components of the compound tensors are related by HOOKE's law for the compound medium (the "effective" medium). The four (real or thought-) experiments can be used to determine the five stiffnesses of the compound medium. However, in this article this concept is used only as the starting point for a general algorithm.

The argument in Bruggeman (1937), Backus (1962), Schoenberg & Muir (1989) and in the present article is based on the observation that in a stack of layers with welded contact with the normal in 3-direction the strain components ε_{11}, ε_{22} and ε_{12} - those which have no subscript "3" - are continuous across the interfaces between the layers, while ε_{33}, ε_{23} and ε_{13} are discontinuous across the interfaces. Similarly, the stress components σ_{33}, σ_{23} and σ_{13}- those with at least one subscript "3" - are continuous across the interfaces between the layers, while σ_{11}, σ_{22} and σ_{12} are discontinuous across the interfaces.

If the continuous components are homogeneous in each constituent layer, they are *fixed* throughout the compound medium, i.e., they are identical to the corresponding effective components. The discontinuous components *vary* from constituent to constituent. The "effective" counterparts of the variable constituent components are their thickness-weighted averages.

A concise algorithm for the elastic stiffnesses of layered media

HOOKE's Law for anisotropic media can be represented in the form

$$\mathbf{I} \cdot \sigma = \mathbf{C} \cdot \varepsilon , \qquad (1)$$

where

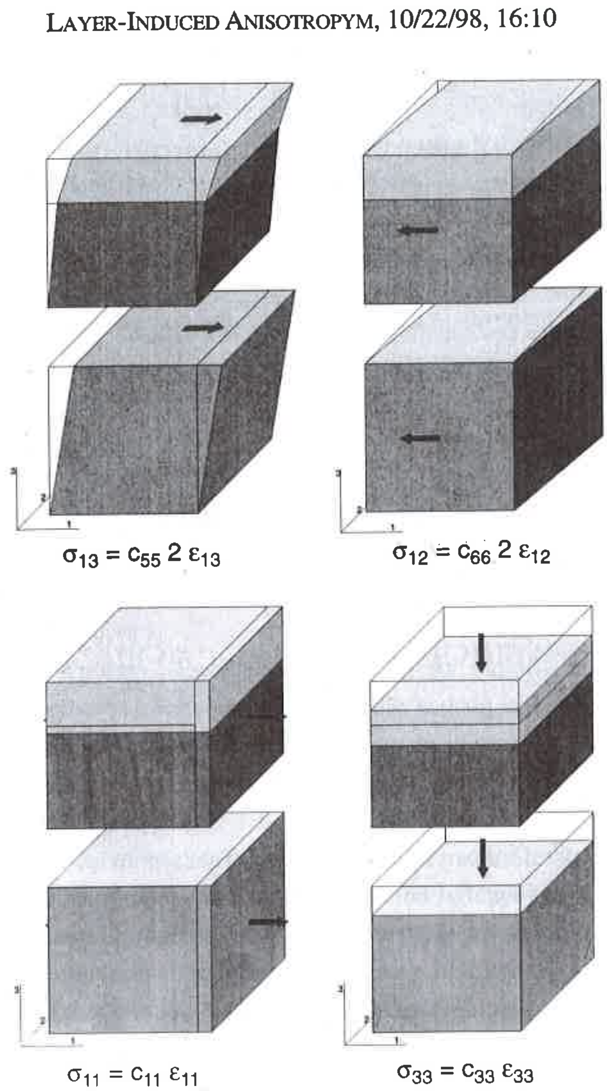

Figure 2 - Four typical stress-strain states of the layer sequence (above) and the effective medium (below) that allows the determination of c_{55}, c_{66}, c_{11}, and c_{33}. The remaining stiffness c_{13} can be determined from the condition that in the last two situation there is no "cross-extension"(contraction). The stress-strain relations below the figures refer to the compound medium.

Figura 2 *- Acima, quatro estágios de esforço/deformação de camadas e, abaixo, o meio efetivo que permite a determinação de* c_{55}, c_{66}, c_{11}, c_{33}. *A rigidez restante* c_{13} *pode ser determinada da condição na qual nas duas últimas situações não existe "extensão cruzada" (contração). As relações de esforço/deformação abaixo das figuras dizem respeito ao meio composto.*

$$\varepsilon = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{pmatrix} = \begin{pmatrix} \varepsilon_{1} \\ \varepsilon_{2} \\ \varepsilon_{3} \\ \varepsilon_{4} \\ \varepsilon_{5} \\ \varepsilon_{6} \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \sigma_{1} \\ \sigma_{2} \\ \sigma_{3} \\ \sigma_{4} \\ \sigma_{5} \\ \sigma_{6} \end{pmatrix}, \qquad (2)$$

$\mathbf{C} = c_{ij}$ and $\boldsymbol{I} = \delta_{ij}$ are the 6x6 stiffness and identity matrices, repectively.

Continuity of stresses across the interfaces requires that the following components:

$$\varepsilon_1\ (=\varepsilon_{11}),\ \varepsilon_2\ (=\varepsilon_{11}),\ \varepsilon_6\ (=2\ \varepsilon_{12}) \qquad (3)$$

are the same throughout the stack of layers. Similarly, continuity of stresses requires that the following components:

$$\sigma_3\ (=\sigma_{33}),\ \sigma_4\ (=\sigma_{23}),\ \sigma_5\ (=\sigma_{13}) \qquad (4)$$

are the same throughout the stack of layers. Therefore, the arrays **f** and **v** of 'fixed' and 'variable' components are introduced:

$$\mathbf{f} = \begin{pmatrix} \varepsilon_{1} \\ \varepsilon_{2} \\ \sigma_{3} \\ \sigma_{4} \\ \sigma_{5} \\ \varepsilon_{6} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} \sigma_{1} \\ \sigma_{2} \\ \varepsilon_{3} \\ \varepsilon_{4} \\ \varepsilon_{5} \\ \sigma_{6} \end{pmatrix} \qquad (5)$$

By re-arranging the variable (**v**-) components and the continuous (**f**-) components in HOOKE's Law (Eq.(1)), one can write:

$$L(\mathbf{C})\,\mathbf{v} = R(\mathbf{C})\,\mathbf{f}, \qquad (6)$$

where $L(\bullet)$ and $R(\bullet)$ are (non-linear) operators on the set of 6x6 matrices. Instead of a general definition, the operators are defined by example:

$$L(\mathbf{C}) = \begin{pmatrix} 1 & 0 & -c_{13} & -c_{14} & -c_{15} & 0 \\ 0 & 1 & -c_{23} & -c_{24} & -c_{25} & 0 \\ 0 & 0 & -c_{33} & -c_{34} & -c_{35} & 0 \\ 0 & 0 & -c_{34} & -c_{44} & -c_{45} & 0 \\ 0 & 0 & -c_{35} & -c_{45} & -c_{55} & 0 \\ 0 & 0 & -c_{36} & -c_{46} & -c_{56} & 1 \end{pmatrix},$$

$$R(\mathbf{C}) = \begin{pmatrix} c_{11} & c_{12} & 0 & 0 & 0 & c_{16} \\ c_{12} & c_{22} & 0 & 0 & 0 & c_{26} \\ c_{13} & c_{23} & -1 & 0 & 0 & c_{36} \\ c_{14} & c_{24} & 0 & -1 & 0 & c_{46} \\ c_{15} & c_{25} & 0 & 0 & -1 & c_{56} \\ c_{16} & c_{25} & 0 & 0 & 0 & c_{66} \end{pmatrix}, \qquad (7)$$

Note that $R(\mathbf{C}) - L(\mathbf{C}) = \mathbf{C} - \mathbf{I}$, and that

$$det\,(R(\mathbf{C})) = -\begin{vmatrix} c_{33} & c_{34} & c_{35} \\ c_{34} & c_{44} & c_{45} \\ c_{35} & c_{45} & c_{55} \end{vmatrix}, \qquad (8)$$

i.e., it is the negative of a principal minor of the original elastic matrix **C**. Since **C** is by definition non-singular, all its principal minors are larger than zero (Ayres, 1962). Thus $L(\mathbf{C})$ is invertible, and Eq.(6) can be recast as:

$$\mathbf{v} = \mathbf{Q} \cdot \mathbf{f}, \text{ with } \mathbf{Q} = [L(\mathbf{C})]^{-1} R(\mathbf{C}). \qquad (9)$$

Note that a re-arrangement similar to that in going from Eq.(1) to Eq.(6) can be applied to Eq.(9) to yield:

$$L(\mathbf{Q})\,\sigma = R(\mathbf{Q})\,\varepsilon, \qquad (10)$$

If $\det(L(\mathbf{Q})) \neq 0$, which can be shown along the same lines as before, Eq.(10) can be recast as

$$\sigma = [L(\mathbf{Q})]^{-1} R(\mathbf{Q})\,\varepsilon, \qquad (11)$$

and by comparison with Eq.(1) one has:

$$\mathbf{C} = [L(\mathbf{Q})]^{-1} R(\mathbf{Q}) = [L\,([L(\mathbf{C})]^{-1} R(\mathbf{C}))]^{-1} R([L(\mathbf{C})]^{-1} R(\mathbf{C})) \qquad (12)$$

This is more than a roundabout way of writing the stiffness matrix. Consider HOOKE's law in the alternative form of Eq.(9) written out for each of the n constituents of a periodic stack of layers:

$$\mathbf{v}^{(j)} = \mathbf{Q}^{(j)}\mathbf{f}^{(j)}, \; j=1,\ldots,n, \qquad (13)$$

or, since the fixed components of **f** are constant throughout the stack,

$$\mathbf{v}^{(j)} = \mathbf{Q}^{(j)}\mathbf{f}. \qquad (14)$$

The thickness-weighted average of the components of **v** are the corresponding effective components for the compound medium:

$$\mathbf{v}^{(eff)} = \langle \mathbf{v}^{(j)} \rangle = \frac{\sum h_j \mathbf{v}_j}{\sum h_j}, \qquad (15)$$

thus by taking the thickness-weighted average on both sides of Eq.(14) one gets:

$$\mathbf{v}^{(eff)} = \langle \mathbf{v}^{(j)} \rangle = \langle \mathbf{Q}^{(j)} \rangle\,\mathbf{f} = \mathbf{Q}^{(eff)}\,\mathbf{f}^{(eff)}. \qquad (16)$$

The terms in the matrix **Q** are the correct arguments for the thickness-weighted averages. Depending on their position in the matrix, they are either stiffnesses, compliances, or dimensionless. To obtain the effective *stiffness* matrix, the operators L and R have to be used again, as described in connection with Eq.(11):

$$\sigma^{(eff)} = [L(\mathbf{Q}^{(eff)})]^{-1} R(\mathbf{Q}^{(eff)})\,\varepsilon^{(eff)},$$

$$\mathbf{C}^{(eff)} = [L(\mathbf{Q}^{(eff)})]^{-1} R(\mathbf{Q}^{(eff)}) = [L\,(\langle [L(\mathbf{C})]^{-1} R(\mathbf{C}) \rangle)]^{-1} R(\langle [L(\mathbf{C})]^{-1} R(\mathbf{C}) \rangle). \qquad (17)$$

The validity of effective stiffnesses for wave propagation

In deriving the effective stiffness matrix, it was assumed that the continuous components (combined in the array **f**) are *constant* throughout the compound medium. The effective stiffness matrix thus applies, in principle, only to infinitely long wave length. This is acknowledged by terminology like "quasi-static stiffness matrix", or "long-wavelength equivalent medium" (Backus, 1962), and "... for wavelengths sufficiently exceeding the thickness of the layer period". It has also been said that the components of f are "slowly varying" and those of **v** "rapidly varying". This is acceptable for qualitative use, but to give these expressions a quantitative meaning, one has to check the approximation inherent in the effective-medium approach against rigorous solutions of the wave equation for periodically layered media. If this is not done, one faces the dilemma that the effective medium theory is valid asymptotically for infinitely long wavelengths, and the ray theory in which one wants to use the effective parameters is valid asymptotically for infinitely small wavelengths.

An analytic solution of the wave equation for a periodic stack of layers was given by Rytov (1956). Four of the five effective stiffnesses of his solution were asymptotically equal to the quasi-static solution. It is likely that the fifth stiffness would have agreed also after careful recalculation

(Rytov had to carry out his calculations by hand), but no such a recalculation seems to have been published. Rytov (1956) gave, qualitatively, as validity limit the order of (tan (kH) - kH), where k is the wave number and H the thickness of one period of the stack. This expression evaluates to 1% for $kH \approx 0.32$, i.e., for $H \approx 0.05\ \lambda$.

A similar calculation was presented by Schoenberg and Gilbert at the 45th EAEG meeting in Oslo. Nothing was published except the abstract (Schoenberg & Gilbert 1983). Helbig (1984) investigated the problem for the propagation of SH waves. In his examples, the approximations agreed within a few percent for $H \approx 0.5\ \lambda$. The level of deviation depends on the particular combination of constituents and on the angle of aperture, so that a general statement can hardly be expected. However, for most situations one should be on safe grounds for $H < 0.15\ \lambda$.

Another aspect that might cast the validity of the effective-medium approach in doubt is the assumption of periodicity, since truly periodic layer sequences are highly unlikely. However, this problem is easily solved by observing that a *compound layer* of thickness H behaves precisely as a layer of the replacement medium with this thickness. The corresponding thought experiment is illustrated in Fig. 3. The response of a layer sequence intercalated between two half spaces cannot be distinguished from that of an intercalated homogeneous replacement layer with the effective parameters, if the probing wavelength is "sufficiently long" (about six times the layer thickness).

This is particularly significant for problems like "blocking of continuous velocity logs". The resolution of the CVL is about 30 cm, while a typical seismic wavelength is 60 m ($f \approx 50$ Hz at $v \approx 3000$ m/s). A depth range of 1000 m thus corresponds to 3300 (presumably isotropic) layers with two parameters each. For the calculation of synthetic seismograms it would be sufficient to have individual (compound) layers of a thickness of 10 m. The replacement of log sections of about 10 m by constant values is called "blocking". However, simple averaging is not sufficient, since contrary to expectations the synthetic seismogram changes with the sampling step. The correct compound layers (or "blocks") can be determined with the algorithm just described. If P- and S-velocities are available, the process results in a sequence of 100 transversely isotropic layers with five parameters each, a reduction in complexity (measured by the number of parameters to be carried) by a factor or 12.5. If only P-wave data are available, only vertically propagating P-waves can be modeled. Blocking as described above reduces the 3300 velocity data to a mere 100.

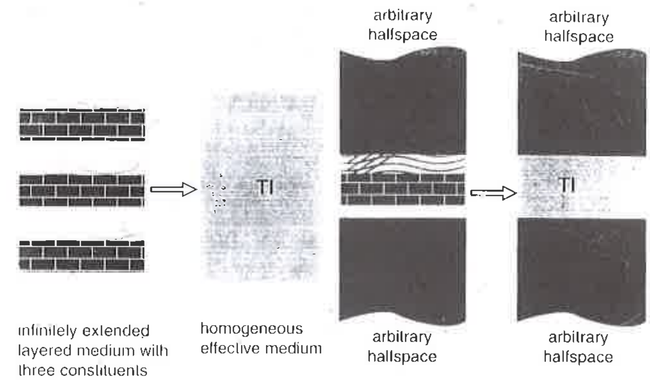

Figure 3 - Left: an infinitely extended periodically layered medium is "long-wave equivalent" to a homogeneous transversely anisotropic "effective medium". Right: a single period of the layer sequence between two arbitrary half spaces is "long-wave equivalent" to a layer of the effective medium.

***Figura 3** - Na esquerda, um meio estratificado, periódico e infinito, tem "equivalência de onda longa" a um "meio efetivo" homogêneo de anisotropia transversa. Na direita, o período simples de uma seqüência de camada entre dois meio-espaços arbitrários tem "equivalência de onda longa" a uma camada de meio efetivo.*

Examples of effective media

The algorithm described is easily implemented as a program. As a test, a MATHEMATICA function was written that directly determines $\mathbf{Q}(\mathbf{C}) = [L(\mathbf{C})]^{-1} R(\mathbf{C})$ for each of the constituent layers. The same function applied to the *effective* $\mathbf{Q}$ - the thickness-weighted average of the individual $\mathbf{Q}$ - gives the effective stiffness matrix of the compound medium. This process allows - within the limits of validity discussed in the previous section - to determine the effective replacement medium corresponding to periodic sequence of layers.

The MATHEMATICA function can be similarly used to determine analytic expressions in terms of symbolic stiffness. For instance, for transversely isotropic constituents with vertical axis of symmetry one obtains for the matrix $\mathbf{Q}^{(TI)}$

$$\mathbf{Q}^{(\mathrm{TI})}=\begin{pmatrix} c_{11}-\frac{c_{13}^2}{c_{33}} & c_{11}-\frac{c_{13}^2}{c_{33}}-2\,c_{66} & \frac{c_{13}}{c_{33}} & 0 & 0 & 0 \\ c_{11}-\frac{c_{13}^2}{c_{33}}-2\,c_{66} & c_{11}-\frac{c_{13}^2}{c_{33}} & \frac{c_{13}}{c_{33}} & 0 & 0 & 0 \\ -\frac{c_{13}}{c_{33}} & -\frac{c_{13}}{c_{33}} & \frac{1}{c_{33}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{c_{55}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c_{55}} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix} \quad (18)$$

Application of the function to $\mathbf{Q}^{(eff)} = <\mathbf{Q}^{(i)}>$, the thickness-weighted average of the individual $\mathbf{Q}$, yields the effective stiffness matrix for the compound medium:

$$\begin{aligned} c_{11}^{(\mathrm{eff,TI})} &= \left\langle c_{11}-\frac{c_{13}^2}{c_{33}}\right\rangle + \left\langle\frac{c_{13}}{c_{33}}\right\rangle^2 \Big/ \left\langle\frac{1}{c_{33}}\right\rangle \\ c_{13}^{(\mathrm{eff,TI})} &= \left\langle\frac{c_{13}}{c_{33}}\right\rangle \Big/ \left\langle\frac{1}{c_{33}}\right\rangle \\ c_{33}^{(\mathrm{eff,TI})} &= 1 \Big/ \left\langle\frac{1}{c_{33}}\right\rangle \\ c_{55}^{(\mathrm{eff,TI})} &= 1 \Big/ \left\langle\frac{1}{c_{55}}\right\rangle \\ c_{66}^{(\mathrm{eff,TI})} &= \langle c_{66}\rangle \end{aligned} \quad (19)$$

whith $c_{12}^{(\mathrm{eff,TI})} = c_{11}^{(\mathrm{eff,TI})} - 2\, c_{66}^{(\mathrm{eff,TI})}$

The process has been applied to sequences of lower symmetry. Though the numerical application is straight forward, the symbolic results are rather complicated. As an example, the intermediate matrix $\mathbf{Q}$ for a sequence of monoclinic layers with parallel two-fold axes perpendicular to the interfaces is given. Even this case is too complicated for display, unless all constituents are rotated about the 3-axis to make $c_{45} = 0$.

$$\mathbf{Q}^{(\mathrm{Mono},3,c_{45}=0)}=\begin{pmatrix} c_{11}-\frac{c_{13}^2}{c_{33}} & c_{12}-\frac{c_{13}c_{23}}{c_{33}} & \frac{c_{13}}{c_{33}} & 0 & 0 & c_{16}-\frac{c_{13}c_{36}}{c_{33}} \\ c_{12}-\frac{c_{13}c_{23}}{c_{33}} & c_{22}-\frac{c_{23}^2}{c_{33}} & \frac{c_{23}}{c_{33}} & 0 & 0 & c_{26}-\frac{c_{23}c_{36}}{c_{33}} \\ -\frac{c_{13}}{c_{33}} & -\frac{c_{23}}{c_{33}} & \frac{1}{c_{33}} & 0 & 0 & -\frac{c_{36}}{c_{33}} \\ 0 & 0 & 0 & \frac{1}{c_{44}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c_{55}} & 0 \\ c_{16}-\frac{c_{13}c_{36}}{c_{33}} & c_{26}-\frac{c_{23}c_{36}}{c_{33}} & \frac{c_{36}}{c_{33}} & 0 & 0 & c_{66}-\frac{c_{36}^2}{c_{33}} \end{pmatrix}. \quad (20)$$

The generalized algorithm is transparent and instructive, but its significance in connection with geological media is perhaps not as large as in connection with industrial compound media.

For geophysical problems the most significant application remains layering of isotropic constituents. It may be instructive to obtain the stiffness directly from the expressions for transversely isotropic constituents with the specification:

$$\begin{aligned} c_{11}^{iso} &= c_{22}^{iso} = c_{33}^{iso} = \frac{\mu}{\theta} \\ c_{23}^{iso} &= c_{13}^{iso} = c_{12}^{iso} = \frac{\mu}{\theta} - 2\mu\,, \\ c_{44}^{iso} &= c_{55}^{iso} = c_{66}^{iso} = \mu \end{aligned} \quad (21)$$

where μ is the shear modulus and θ is the ratio $(v_S/v_P)^2$.

One obtains for the components of the effective stiffness matrix of a medium consisting of isotropic layers as:

$$\begin{aligned} c_{11}^{\mathrm{TI,isol.}} = c_{22}^{\mathrm{TI,isol.}} &= \langle 4\mu(1-\theta)\rangle + \frac{\langle(1-2\theta)^2\rangle}{\langle\theta/\mu\rangle} \\ c_{33}^{\mathrm{TI,isol.}} &= \frac{1}{\langle\theta/\mu\rangle} \\ c_{13}^{\mathrm{TI,isol.}} = c_{23}^{\mathrm{TI,isol.}} &= \frac{\langle(1-2\theta)\rangle}{\langle\theta/\mu\rangle} \\ c_{44}^{\mathrm{TI,isol.}} = c_{55}^{\mathrm{TI,isol.}} &= \frac{1}{\langle 1/\mu\rangle} \\ c_{66}^{\mathrm{TI,isol.}} &= \langle\mu\rangle \\ c_{12}^{\mathrm{TI,isol.}} = c_{11}^{\mathrm{TI,isol.}} - 2\,c_{66}^{\mathrm{TI,isol.}} &= \langle 2\mu(1-2\theta)\rangle + \frac{\langle(1-2\theta)^2\rangle}{\frac{1}{\langle\theta/\mu\rangle}} \end{aligned} \quad (22)$$

Two observations can be made immediately by inspection of Eq.(22):

1) If μ is the same throughout the sequence, then $c_{55} = c_{66}$, $c_{11} = c_{33}$, $c_{12} = c_{13}$, i.e., the effective medium is isotropic even if θ varies:

$$\begin{aligned} c_{11}^{\mathrm{TI,isol.},\,\mu=\mathrm{const}} &= 4\mu(1-\langle\theta\rangle) + \frac{\mu(1-2\langle\theta\rangle)^2}{\langle\theta\rangle} = \frac{\mu}{\langle\theta\rangle} \\ c_{33}^{\mathrm{TI,isol.},\,\mu=\mathrm{const}} &= \frac{\mu}{\langle\theta\rangle} \\ c_{13}^{\mathrm{TI,isol.},\,\mu=\mathrm{const}} &= \frac{\mu}{\langle\theta\rangle} - 2\,\mu \\ c_{55}^{\mathrm{TI,isol.},\,\mu=\mathrm{const}} &= \mu \\ c_{66}^{\mathrm{TI,isol.},\,\mu=\mathrm{const}} &= \mu \\ c_{12}^{\mathrm{TI,isol.},\,\mu=\mathrm{const}} &= \frac{\mu}{\langle\theta\rangle} - 2\,\mu \end{aligned} \quad (23)$$

This result might be surprising, but has little significance for seismic applications.

2) If θ, the squared ratio of S- to P-waves, is the same throughout the sequence, then two additional equations hold: $c_{13} = c_{33} - 2\, c_{55}$, and $c_{33}\,(c_{11} - 2\, c_{66}) = c_{13}\,(c_{13} + 2\, c_{66})$, thus there are only three instead of five degrees of freedom. In this and other texts, such media are called K-media. In a K-medium the center of curvature of the wave front of compressional waves in the vicinity of the axis of symmetry lies in the origin (Krey & Helbig, 1956), i.e., it does not differ from that for an isotropic medium. In most geological sections the ratio is constrained to a rather narrow range, so that the conditions for a K-medium are nearly satisfied.

The elastic stiffness for a K-medium are

$$
\begin{aligned}
c_{11}^{K} &= 4(1-\theta)\langle\mu\rangle + \frac{(1-2\theta)^2}{\theta/\langle\mu\rangle}\\
c_{33}^{K} &= \frac{1}{\theta/\langle\mu\rangle}\\
c_{13}^{K} &= \frac{(1-2\theta)}{\theta/\langle\mu\rangle}\\
c_{55}^{K} &= \frac{1}{\langle 1/\mu\rangle}\\
c_{66}^{K} &= \langle\mu\rangle\\
c_{12}^{K} &= 2(1-2\theta)\langle\mu\rangle + \frac{(1-2\theta)^2}{\theta/\langle\mu\rangle}
\end{aligned}
\qquad (24)
$$

The ratio c_{66}/c_{55}, or rather the number $\sqrt{(c_{66}/c_{55})}-1 = v_{SH}/v_{SV} - 1$ expressed in percent, is often called the "shear-wave anisotropy". For layer-induced anisotropy there is no limit to this ratio other that it must be at least unity. However, for reasonable fluctuations of the shear modulus, the number remains modest (compare Fig. 4).

The velocity of P-waves in *horizontal* direction in a K-medium can deviate significantly from the velocity in vertical direction and thus indicate a sizeable "P-wave anisotropy". The fact that over a cone of 30° about the axis of symmetry the wave surface (and the slowness surface) approximate very closely a sphere centered at the origin means that this anisotropy cannot be detected by surface observation of P-waves with standard offsets. Conversely, anisotropy due to lamination with a small fluctuation of q does not influence standard seismic data acquisition and processing. This appears to be the single most important reason why "isotropic" processing and interpretation are, in general, sufficient. Note that over the range of nearly spherical slowness surface also the direction of the P-ray and direction of the P-displacement are nearly that of the normal to the P-front (the slowness direction).

It goes without saying that for "non-standard" acquisition (refraction seismics, VSP with large offset, well-to-well seismics, and particularly shear-wave seismics) the K medium does not afford any simplification.

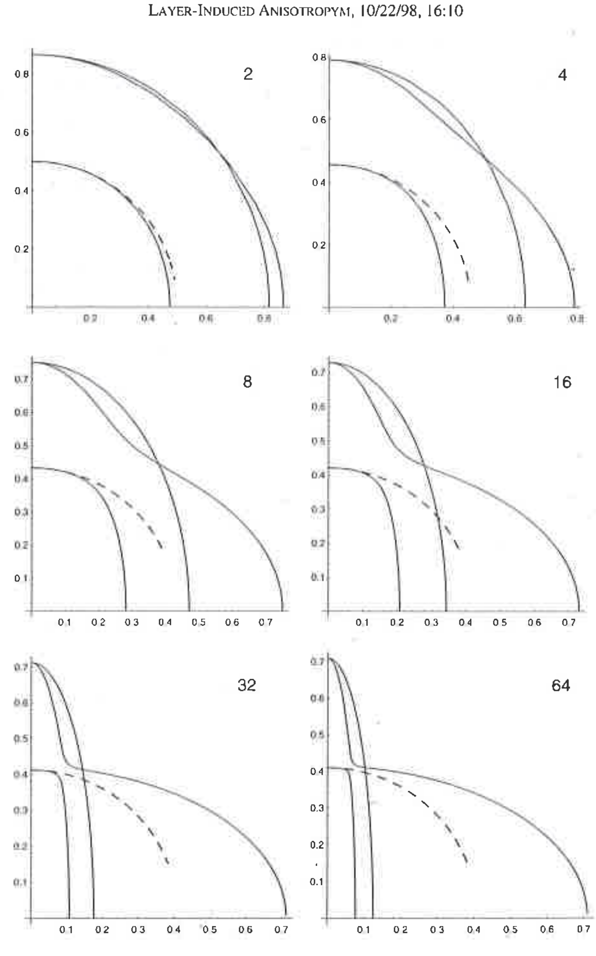

Figure 4 - Slowness surfaces of K-media corresponding to stacks with two equally thick constituents and $\theta = 1/3$. The parameter in the upper right of each panel is the ratio of the two shear stiffnesses.

Figura 4 *- Superfícies de vagarosidade da média-K correspondendo ao empilhamento com dois constituintes de mesma espessura e θ = 1/3. O parâmetro indicado no lado direito superior de cada figura é a razão de duas rigidez de cizalhamento.*

CONCLUSIONS

Any laminated medium, i.e., one that is layered on a scale below the resolution employed, can be replaced by an effective medium which is less heterogeneous than the original medium. If the layering is periodic, the effective medium is homogeneous.

The elastic parameters of the effective compound medium are functions of the elastic parameters of the constituents, but not simple averages. A matrix algorithm is proposed that allows to regard them as "generalized averages": a function is applied to the stiffness matrix, thickness-weighted arithmetic averages are taken, and then the inverse function is applied to the averaged matrix to obtain the stiffness matrix of the effective medium. The process works with any type of constituents and should be of interest to material sciences with respect to laminates as plywood.

Though geological media may be anisotropic even without layering, the most important application of the algorithm in geophysics is based on layering of isotropic constituents. The effective matrix of a compound medium consisting of isotropic constituents has two important properties: if all constituents have the same shear stiffness, the compound medium is isotropic, and if all constituents have the same ratio of S- to P-waves, the behavior of P-waves in the vicinity of the axis does not differ significantly from that in an isotropic medium.

REFERENCES

AYRES, F. jr. -1962- Matrices. Schaum's Outline Series, McGraw-Hill.

BACKUS, G.E. -1962- Long-wave elastic anisotropy produced by horizontal layering. Journal of Geophysical Research, **67:** 4427-4440.

BRUGGEMAN, D.A.G. -1937- Berechnungen der verschiedenen physikalischen Konstanten von heterogenen Substanzen. Teil 3: Die elastischen Konstanten der quasi-isotropen Mischkörper aus isotropen Substanzen. Annalen der Physik, **29(5):** 160-160.

HELBIG, K. -1958- Elastische Wellen in anisotropen Medien. Gerlands Beiträge zur Geophysik, **67:** 177-211; 256-288.

HELBIG, K. -1981- Systematic classification of layer-induced transverse isotropy. Geophysical Prospecting **29:** 550-577.

HELBIG, K. -1984- Anisotropy and dispersion in periodically layered Media. Geophysics **49:** 364-373.

KREY, Th. & HELBIG, K. -1956- A theorem concerning anisotropy of stratified media and its significance for reflection seismics. Geophysical Prospecting **4:** 294-302.

POSTMA, G.W. -1955- Wave propagation in a stratified medium. Geophysics **20:** 780-806.

RIZNICHENKO, JU. W. -1948- The propagation of seismic waves in discrete and heterogeneous media (in Russian). Isvestiya AN SSSR, ser. geogr. i. geoph. **13:** 115-128.

RIZNICHENKO, JU. W. -1949- On seismic quasi anisotropy (in Russian). Isvestiya AN SSSR, ser. geogr. i. geoph. **13:** 518-544.

RYTOV, S.M. -1956- Acoustical properties of a thinly laminated Medium. Soviet physical Acoustics **2:** 67-80.

SCHOENBERG, M. & Gilbert, K.E. -1983- Elastic waves in periodically layered media. Technical Programme and Abstracts of Papers, 45th Meeting of the European Association of Exploration Geophysicists (Oslo): 58-58

SCHOENBERG, M. & Muir, F. -1989- A calculus for finely layered anisotropic media. Geophysics **54:** 581-589.

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LAYER-INDUCED ELASTIC ANISOTROPY - PART 1: FORWARD RELATIONS BETWEEN CONSTITUENT PARAMETERS AND COMPOUND MEDIUM PARAMETERS

Anisotropy of solids is due to their internal structure, and one of the simplest internal structures is "lamination", i.e., layering on a scale smaller than the wavelengths. Anisotropy due to layering can be analytically described. Typical examples of laminated media are industrial laminates (plywood, resinated paper or textiles as substratum for electronic devices, etc.), and, among the geological media, schists, phyllites, gneiss, clays, and so on). Packages of sedimentary layers can often be regarded as laminated compound media. If such a layering can be approximated by a strictly periodic sequence of layers, the properties of the homogeneous compound medium are those of a single layer-period, a "fundamental sample".

The elastic parameters of a layered compound medium can be expressed in terms of those of the constituents. This is easily done by a sequence of thought experiments, where the *global* stress, and thus the "effective stiffnesses", of the fundamental sample under different *global* strains are determined. However, a more fundamental approach that applies to the entire elastic matrix was already implicit in the first published investigation of the problem (Bruggeman, 1937). A similar idea was independently advanced by Backus (1962), to whom the resulting algorithm is generally attributed.

This idea can be seen as a reformulation of Hooke's law: instead of expressing the components of the stress tensor in terms of the components of the strain tensor, one expresses the "variable" components of both tensors in terms of their "fixed" counterparts, where the fixed components are those that must be continuous across the layer interfaces. The matrix of elastic stiffness is, in this formulation, replaced by a matrix with components of different dimensions (dimensionless or with the dimension of either stiffness or compliance). The modified Hooke's law for the compound medium is obtained by taking the weighted averages of both sides. Since in this formulation the independent variables are constant, the averaging on the r.h.s. applies only to the components of the matrix. This means that matrix components of the constituents are precisely those that can be averaged to obtain the corresponding matrix components of the compound medium.

To obtain the effective components of the genuine stiffness matrix of the compound medium, one has only to re-write Hooke's law for the compound medium in standard form. In simple cases this can be easily done by hand, but in more complicated cases a matrix manipulation program is to be preferred.

The algorithm just described works with *any* type of layers, even triclinic ones. For geophysical applications, the most interesting type of lamination is that with isotropic constituents. Even if individual geological layers can be treated as isotropic, a sequence of layers should be anisotropic. This is indeed so, but for constituents with a narrow range of Poisson ratios the resulting anisotropy does not affect the propagation of compressional waves within a relatively wide aperture (~30°) about the normal to the layers. Thus this type of anisotropy is significant for shear-wave seismics, for refraction seismics, and for well-to-well seismics, but can be neglected in standard reflection seismics.

NOTES ABOUT THE AUTHOR
NOTAS SOBRE O AUTOR

Klaus Helbig

Prof. Klaus Helbig obtained his doctorate in geophysics from the University of Göttingen in 1955. He has been involved in the application of geophysics to engineering problems (Assuan dam, Turbela dam) and to the exploration for oil, gas, and iron ores since 1952. He taught at universities in Germany, Pakistan, the United States, Belgium, The Netherlands, Israel, Indonesia, Australia, and France. For 24 months (spread over four years) he acted as Unesco Expert, teaching Geophysics at the University of the Panjab, Lahore, Pakistan. Full-time employment was with industry (Seismos GmbH, Texaco Inc, Texaco Belgium, Deutsche Texaco) and various universities (Munich, St. Louis). The last was a full professorship at the University of Utrecht combined with an associate professorship at the City University of Amsterdam (later at the Free University of Amsterdam), both in Exploration Geophysics, from 1977 to 1992.

He has published about 75 papers, translated one book, and written a book on Seismic Anisotropy. He is co-author of a book on High Resolution Seismics published in 1998.

His editing career began as co-editor of the Geological Bulletin of the University of the Panjab in 1961. From 1969 to 1988 he was editor of Geophysical Prospecting, for the last years simultaneously Editor=ADin=ADChief for the European Association of Exploration Geophysicists. He is one of the two Series Editors of the Handbook of Exploration Geophysics (Pergamon imprint of Elsevier Science). During his editing career he edited about 1500 journal articles, fifteen books in the handbook series and three books published by various scientific organizations. He joined Editerra in 1969 and was president for one three-year term in the sixties.

He has participated in numerous workshops, mainly on seismic anisotropy. Among these was the Second Brazilian Workshop on Seismic Anisotropy (November 1996). The current paper is based on his invited contribution.

Since his retirement from the University of Utrecht he has held advisory positions in the Netherlands, with the Commission of the European Communities, and with the Osservatorio Geofisico Sperimentale in Trieste. He was the Expert of the European Commission for the Joule I project, where he supervised and coordinated the research of about 100 European scientists from most member states of the European Union. As a visiting professor he worked at International School of Applied Geophysics (Erice, Sicily, three times), the University of Uppsala (two times), at Schlumberger Doell Research (several times), the University of Tel Aviv, Gadja Mahda University (Jogjakarta, Indonesia), 13 Australian Universities during a tour as Esso Distinguished Professor, Texaco Research Center (Houston), the Institut Francais du Petrole, the University if Pau, and the Osservatorio Geofisico Sperimentale in Triest. At the time of writing(January, 1999), he is preparing for a three-month stay at the Centro de Geociências, Universidade Federal do Pará, Belém, expected to begin in March 1999. Currently, he is a member of the American Geophysical Union, the Society of Exploration Geophysicists (honorary member since 1982), the European Association of Exploration Geophysicists (honorary member since 1989), the European Section of the environmental and Engineering Geophysical Society, the German Geophysical Society, the (Dutch) Royal Society for Geology and Mining, and the European Association of Science Editors.