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CRITERIA FOR ESTIMATING ELASTIC PARAMETERS BY SINGULAR VALUE DECOMPOSITION

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ABSTRACT. The analysis of seismic amplitudes has made the use of the seismic method in oil exploration more successful. This is due to the fact that amplitude anomalies can bring valuable information about the presence of hydrocarbons in the pores of a rock. Several studies have been carried out to analyze and establish criteria for such amplitude anomalies, thus giving rise to the Amplitude versus Offset (AVO) technique. This work aims to obtain elastic parameters of subsurface rocks through generalized inversion by the method of singular value decomposition (SVD). Based on the study of eigenvalues and eigenvectors through computational modeling with SVD, an analysis was made to obtain the elastic parameters and the criteria that must be used to obtain a coherent estimate in the inversion process. Two approximations by Aki and Richards were used, and compared with other approximations presented in the literature. In all of the studied approximations, the first parameter ($\Delta v_P/v_P$ or $\Delta Z_P/Z_P$, where v_P is the P-wave velocity and Z_P is the P-wave impedance) was the one with the lowest percentage error and, depending on the maximum angle of incidence, the estimated value coincides with the exact one. The second parameter ($\Delta v_S/v_S$ or $\Delta Z_S/Z_S$ where v_S is the S-wave velocity) was recovered satisfactorily in some cases and the third parameter ($\Delta \rho/\rho$, where ρ is the density) was not recovered satisfactorily.

Keywords: exploration seismology, seismic inversion, amplitude versus offset, singular value decomposition.

RESUMO. A análise das amplitudes sísmicas tornou a utilização do método sísmico na exploração de petróleo mais bem sucedida. Isso se deve ao fato de que anomalias de amplitudes podem trazer informações valiosas sobre a presença de hidrocarbonetos nos poros de uma rocha. Diversos estudos foram feitos para analisar e estabelecer critérios para tais anomalias de amplitude, surgindo assim a técnica Amplitude *versus* Afastamento (AVO). Este trabalho tem como objetivo a obtenção de parâmetros elásticos das rochas de subsuperfície através da inversão generalizada pelo método da decomposição por valores singulares (SVD). Com base no estudo dos autovalores e autovetores através da modelagem computacional com SVD, foi feita uma análise na obtenção dos parâmetros elásticos e os critérios que devem ser utilizados para obter uma estimativa coerente no processo de inversão. Foram empregadas duas aproximações de Aki e Richards e comparadas com outras aproximações apresentadas na literatura. Em todas as aproximações estudadas, o primeiro parâmetro ($\Delta v_P/v_P$ ou $\Delta Z_P/Z_P$, sendo v_P a velocidade da onda P e Z_P a impedância da onda P) foi o que apresentou o menor erro percentual e, a depender do angulo máximo de incidência, o valor estimado coincide com o exato. O segundo parâmetro ($\Delta v_S/v_S$ ou $\Delta Z_S/Z_S$, sendo v_S a velocidade da onda S) foi recuperado satisfatoriamente em alguns casos e o terceiro parâmetro ($\Delta \rho/\rho$, sendo ρ a densidade) não foi recuperado satisfatoriamente.

Palavras-chave: sismologia de exploração, inversão sísmica, amplitude versus afastamento, decomposição por valores singulares.

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INTRODUCTION

Exploration seismology is the branch of Geophysics that studies the propagation of elastic waves within the Earth with the aim of imagining the subsurface. The success of the seismic method in the exploration of hydrocarbons has become unquestionable and new tools have emerged and improved the method, making its use increasingly effective. Among these techniques, the amplitude versus offset analysis (AVO) stands out, which is the study of the variation of the reflected amplitude with the source-receiver distance. AVO was introduced by Ostrander (1984), when he demonstrated that the reflection coefficient of a sandstone with gas varies anomalously with increasing spacing, so that it could be used as a direct indicator of hydrocarbons. AVO analysis results from the theoretical relationship between the reflection coefficient, the wave incidence angle, the variations in P-wave and S-wave velocities and the variations in density through an interface. However, the exact expression that defines the reflection coefficient is very complex, making it necessary to formulate approximate and linear equations for it, proposed by several authors. The purpose of these approaches is to obtain a better understanding of the elastic parameters and to simplify their determination.

More recently, Ma & Sun (2019) performed a direct inversion of Young's modulus and Poisson's ratio using exact Zoeppritz equations, and Liang et al. (2017) studied the stability study of pre-stack seismic inversion based on the full Zoeppritz equation.

Starting from the AVO concept, in this study we used the singular value decomposition (SVD) technique to obtain the elastic parameters of the rocks, expressed by approximations of the exact equations of the seismic reflection coefficient, called Zoeppritz equations. We have studied, in particular, the Aki and Richards (2002) approximation, comparing it with the approximations presented by other authors, such as Bortfeld (1961), Shuey (1985) and Thomsen (1990). We discuss the validity and precision limits for each approximation using graphs. The tests were made by studying three models studied by Ostrander (1984), fixing values of the Poisson's ratio and changing the contrasts of P-wave velocity and density.

Using SVD, we verified which parameters of Aki and Richards (2002) approximation can best be obtained in the inversion process, taking into account the system of linear equations conditioning through the analysis of eigenvalues and eigenvectors. Finally, we used the reflection coefficient values as input data to obtain the elastic parameters. The inversion was made with synthetic data corresponding to a gas-filled reservoir, which is the model used by Ostrander (1984).

Amplitude versus Offset

The reflection coefficient of a P-wave is defined as the ratio between the amplitudes of the reflected and incident P-waves, and it depends on the P-wave velocity, the S-wave velocity and the density of each of the layers that define an interface. The reflection coefficient for normal incidence is given by (Aki and Richards, 2002):

$$R_0 = \frac{A_1}{A_0} = \frac{v_{P2}\rho_2 - v_{P1}\rho_1}{v_{P2}\rho_2 + v_{P1}\rho_1}, \quad (1)$$

where A_0 and A_1 are the amplitudes of the incident and reflected P-waves, respectively; v_{P1} and v_{P2} are the P-wave velocities of media 1 and 2, respectively; and ρ_1 and ρ_2 are the densities of media 1 and 2, respectively. In equation (1), for both media, the product between density and P-wave velocity is called acoustic P-wave impedance, denoted by Z_P.

The amplitude analysis of the reflected wave as a function of the incidence angle can be used to detect lateral variations in the elastic properties of reservoir rocks. One such property is the Poisson's ratio. The expressions for the reflection coefficients of plane waves in terms of the amplitudes as a function of the incidence angle were developed by Zoeppritz in 1919 (Aki and Richards, 2002:

$$\begin{pmatrix} \cos\theta_1 & \frac{v_{P1}}{v_{S1}}\mathrm{sen}\phi_1 & \frac{v_{P1}}{v_{P2}}\cos\theta_2 & \frac{v_{P1}}{v_{S2}}\mathrm{sen}\phi_2 \\ -\mathrm{sen}\theta_1 & \frac{v_{P1}}{v_{S1}}\cos\phi_1 & \frac{v_{P1}}{v_{P2}}\mathrm{sen}\theta_2 & \frac{v_{P1}}{v_{S2}}\cos\phi_2 \\ -\cos 2\phi_1 & -\mathrm{sen}2\phi_1 & \frac{\rho_2}{\rho_1}\cos 2\phi_2 & \frac{\rho_2}{\rho_1}\mathrm{sen}2\phi_2 \\ \mathrm{sen}2\theta_1 & -\frac{v_{P1}^2}{v_{S1}^2}\cos 2\phi_1 & \frac{\rho_2 v_{P1}^2 v_{S2}^2}{\rho_1 v_{P2}^2 v_{S1}^2}\cos 2\theta_2 & \frac{\rho_2 v_{P1}^2}{\rho_1 v_{S1}^2}\cos 2\phi_2 \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 \\ \mathrm{sen}\theta_1 \\ \cos 2\phi_1 \\ \mathrm{sen}2\theta_1 \end{pmatrix}, \quad (2)$$

where v_{S1} and v_{S2} are the S-wave velocities in media 1 and 2, respectively; θ_1, ϕ_1, θ_2 and ϕ_2 are, respectively, the incidence angles of the P- and S-waves, and the reflected angles of the P- and S-waves; A_1, B_1, A_2 and B_2 are, respectively, the amplitudes of the incident P- and S-waves, and the amplitudes of the reflected P- and S-waves.

Due to their relative complexity and non-linearity, Zoeppritz's equations do not allow an easy identification of the petrophysical parameters that influence the behavior of the reflection coefficients. However, the greatest interest of seismic exploration is in the relationship between an incident P-wave and its reflection as a function of incidence angle, given by the coefficient A_1, as well as in the estimation of elastic parameters of reservoir rocks, relating these parameters to the fluid contained in these reservoirs. The analysis of elastic parameters using Zoeppritz's equations becomes easier through linearized approximations of the exact solution such as, for example, the expression proposed by Aki and Richards (2002).

The AVO technique is the study of the amplitude variation in the reflected wave in relation to the source-receiver distance, and it is used in seismic reflection to infer certain characteristics of the rocks. It was introduced by Ostrander (1984), who demonstrated that the reflection coefficient of a sandstone containing gas varies anomalously in relation to distance increase. Thus, this type of behavior can be used as a direct indicator of hydrocarbons. The recorded amplitude generally decreases with the distance, depending on the shear velocity. And at the base of siliciclastic reservoirs it usually increases with distance (Simm and Bacon, 2014). However, in the presence of gas, the opposite occurs, that is, there is an anomalous growth.

The AVO analysis results from the theoretical relationship between the reflection coefficient and: (i) the incidence angle or distance (offset), (ii) P-wave velocity, (iii) S-wave velocity, (iv) the density variations through an interface (Ostrander, 1984). This analysis has been used with considerable success to indicate the presence of hydrocarbons. Furthermore, understanding the interrelationship between seismic properties and physical characteristics of the environment such as lithology, porosity and fluid content in the pores of rocks is a necessary information for the quantitative extraction of information using the AVO technique (Ostrander, 1984).

Koefoed (1955) investigated the variation of the reflection amplitude with respect to the incidence angle, and the use of this analysis as an indicator of the variation of the v_P/v_S ratio. He considered three elastic properties of each medium: P-wave velocity, density and Poisson's ratio. He also used simplifications of Zoeppritz's equations, defining four parameters that govern the behavior of the reflection coefficient between two isotropic media: (i) the P-wave velocities ratio; (ii) the density ratio; (iii) the Poisson's ratio of the upper layer and (iv) the Poisson's ratio of the lower layer.

Ostrander (1984) demonstrated how the reflection coefficients influence the behavior of seismic wave amplitudes according to the four parameters mentioned above. The conditions imposed are restricted to regions with small angles and smaller than the pre-critical angle. In general, if the impedance and the Poisson's ratio both increase or decrease from the upper to the lower layer, the reflection coefficient modulus and the absolute amplitude increase with the source-receiver distance and there is an AVO anomaly. If the impedance and Poisson's ratio behave differently, the reflection coefficient modulus is decreasing and there is a decrease of amplitude with the distance. These conclusions obtained by Ostrander (1984) were based on a proposed model that consisted of a layer of shale interspersed by a layer of sandstone with gas. In most cases, gas-saturated sandstones that produce anomalies in amplitude have low impedance in relation to the shale that surrounds it, as well as reflections that grow in magnitude with distance. The great asset of AVO analysis lies in the relationship between Poisson's ratio and lithology. The presence of gas in pores of rocks causes the P-wave velocity to decrease while the S-wave velocity remains unchanged (Rüger and Gray, 2014).

GENERALIZED INVERSION AND SINGULAR VALUE DECOMPOSITION

The linear inverse problem is formulated as a system of linear equations:

$$\boldsymbol{d} = \boldsymbol{G}\boldsymbol{m}, \quad (3)$$

where $\boldsymbol{d} = [d_1, d_2, \dots, d_M]^T$ is the column vector that represents the observed data, $\boldsymbol{m} = [m_1, m_2, \dots, m_N]^T$ is the column vector that represents the set of model parameters to be estimated and $\boldsymbol{G}$ is the $M \times N$ matrix that relates the vectors $\boldsymbol{d}$ and $\boldsymbol{m}$. In the inverse procedure the unknown is the vector of model parameters $\boldsymbol{m}$, so that the solution of Equation (3) is expressed by (Aster et al., 2016):

$$\boldsymbol{m} = \boldsymbol{G}^{+}\boldsymbol{d}, \quad (4)$$

where $\boldsymbol{G}^{+}$ is a $N \times M$ matrix, called generalized inverse.

A linear inverse problem is considered well-posed or well-conditioned if it satisfies the conditions of existence, uniqueness and stability. The first two conditions require that the system has a solution and that it is unique. The latter condition is more difficult to achieve, since many geophysical inversion problems are unstable. To check whether a given system is well- or ill-conditioned, we determine the condition number (CN) of the matrix, expressed by (Aster et al., 2016):

$$\mathrm{C}N = \frac{\lambda_{\max}}{\lambda_{\min}}, \quad (5)$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalue of the matrix, respectively.

To obtain a generalized inverse or pseudo-inverse matrix, we used singular value decomposition. The matrix $\mathbf{G}$ can be decomposed as:

$$\boldsymbol{G} = \boldsymbol{U}\boldsymbol{\Sigma}V^{T}, \quad (6)$$

Where $\boldsymbol{U}_{M\times M}$ is the matrix that contains the orthonormalized eigenvectors of $\boldsymbol{G}\mathbf{G}^T$; $\boldsymbol{\Sigma}_{M\times N}$ is the diagonal matrix that contains the square root of the eigenvectors of $\boldsymbol{G}^T\boldsymbol{G}$, called singular values; and $\boldsymbol{V}_{N\times N}$ is the matrix that contains the orthonormalized eigenvectors of $\boldsymbol{G}^T\boldsymbol{G}$. The generalized inverse is expressed by (Aster et al., 2016):

$$\boldsymbol{G}^{+} = \boldsymbol{V}\boldsymbol{\Sigma}^{+}\boldsymbol{U}^{T}, \tag{7}$$

where $\boldsymbol{\Sigma}^{+}_{N\times M}$ is the matrix that contains the reciprocal of the non-zero singular values, on the main diagonal of the square part of the matrix.

MAIN APPROXIMATIONS OF THE REFLECTION COEFFICIENT

The exact expression of the reflection coefficient was formulated by Červený et al. (1977) from Zoeppritz's equations. These equations demand the following elastic parameters and information: the P-wave velocities in the upper and lower layers, the S-wave velocities in the upper and lower layers, the density in the upper and lower layers, and the incidence angle.

The set of equations involves four reflection coefficients and, for an incident and reflected P-wave, we have:

$$R_{PP} = [Q^2 + \gamma T_2T_3 + (\gamma - Q)^2T_3T_4 - (1+Q)^2T_1T_2 + (\gamma - 1 - Q)^2T_1T_2T_3T_4]/D, \tag{8}$$

where

$$D = Q^2 + \gamma T_2T_3 + (\gamma - Q)^2T_3T_4 + \gamma T_1T_4 + (1+Q)^2T_1T_2 + (\gamma - 1 - Q)^2T_1T_2T_3T_4,$$

$$Q = 2p^2(\gamma v_{S2}^2 - v_{S1}^2),$$

$$T_1 = pv_{P1}/(1 - p^2v_{P2}^2)^{\frac{1}{2}},$$

$$T_2 = pv_{S1}/(1 - p^2v_{S1}^2)^{\frac{1}{2}},$$

$$T_3 = pv_{P2}/(1 - p^2v_{P2}^2)^{1/2},$$

$$T_4 = pv_{S2}/(1 - p^2v_{S2}^2)^{\frac{1}{2}}, \text{ and}$$

$$\gamma = \frac{\rho_1}{\rho_2}.$$

In Equation (8) ρ_i, v_{Pi}, v_{Si} and p are, respectively, the density, the P-wave velocity, the S-wave velocity, and the ray parameter. The sub-index represents the top layer if it is 1 and the bottom if it is 2. This study will be limited only to the approximations of Equation (8), which defines the PP reflection coefficient. It is important to stress that this approach is valid for a isotropic, viscoelastic medium.

Due to the relative complexity of the exact expression for the reflection coefficient of elastic waves, it became necessary to formulate approximations for it, in order to obtain a better understanding of the influence of elastic parameters and to facilitate their determination using the AVO technique. Barros (1997) and Barros and Ramos (1997) analyze various types of approaches present in the literature, such as Bortfeld (1961), Aki and Richards (2002), Shuey (1985) and Thomsen (1990).

Two steps are necessary in the inversion of AVO data. First, the linearized expressions of the seismic reflection coefficient must be validated, by comparison to the exact expression. Second, we have to analyze the behavior of the eigenvalues and eigenvectors as function of the incidence angle. Although we use the linearized inversion in this study, it is also possible to perform a non-linear approach, without making use of approximate expressions.

BORTFELD APPROXIMATION

Bortfeld (1961) formulated the first approximation of Zoeppritz's equations obtaining the following expression:

$$R_{PP}(\theta_1) \approx \frac{1}{2} ln\left(\frac{v_{P2}\rho_2 \cos\theta_2}{v_{P1}\rho_1 \cos\theta_1}\right) + \left(\frac{ln\frac{\rho_1}{\rho_2}}{ln\frac{v_{P2}}{v_{P1}} - ln\frac{v_{P2}v_{S2}}{v_{P1}v_{S1}}}\right) \mathrm{X}\frac{v_{S1}^2 - v_{S2}^2}{v_{P1}^2}\mathrm{sen}^2\theta_1. \tag{9}$$

The first term of this equation is called the acoustic effect, while the second one is called the elastic effect. This approach is only valid for angles smaller than the critical one. By critical angle we mean an incidence angle that results in a transmission angle equal to 90°. Bortfeld (1961) suggested that the general features of the exact curves are repeated by the approximate curves and that the difference between the exact and approximate values is not greater than a small percentage value. However, Bortfeld (1961) also points out that the mentioned difference increases with the increase of the incidence angle and with the increase in the elastic parameter contrast.

AKI AND RICHARDS APPROXIMATION

Aki and Richards (2002) obtained an approximation for the reflection coefficient $R_{PP}(\theta)$ deriving the first order effect of small variations in density and P-wave and S-wave velocities for the problem of an interface between two solids. The objective was to verify the separate contributions of the density variation ($\Delta\rho$), the P-wave velocity variation (Δv_P), and the S-wave velocity variation (Δv_S). The approximation was derived from the exact formula, replacing each of the parameters of the two media by its average values and their differences. The considered angle is the average of the incidence and transmission angles. The obtained equation is given by:

$$R_{PP}(\theta) \approx \frac{1}{2\cos^2\theta}\left(\frac{\Delta v_P}{v_P}\right) + \frac{1}{2}\left[1 - 4\left(\frac{v_S}{v_P}\right)^2 \operatorname{sen}^2\theta\right]\left(\frac{\Delta\rho}{\rho}\right) - 4\left(\frac{v_S}{v_P}\right)^2 \operatorname{sen}^2\theta\left(\frac{\Delta v_S}{v_S}\right), \quad (10)$$

where

$$\Delta v_P = v_{P2} - v_{P1},$$

$$v_P = \frac{v_{P1} + v_{P2}}{2},$$

$$\Delta v_S = v_{S2} - v_{S1},$$

$$v_S = \frac{v_{S1} + v_{S2}}{2},$$

$$\Delta\rho = \rho_2 - \rho_1,$$

$$\rho = \frac{\rho_1 + \rho_2}{2},$$

$$\theta = \frac{\theta_1 + \theta_2}{2}.$$

This equation presents three explicit parameters to be obtained in the inversion process. The parameter $\Delta v_P/v_P$ describes the variation in the P-wave velocity and can be defined as its reflectivity. The analogue applies to the parameter$\Delta v_S/v_S$. The relative density variation is $\Delta\rho/\rho$.

The P- and S-wave reflectivity can provide valuable information about fluid changes in reservoirs, such as sandstones filled with gas and salt water, for example.

SHUEY'S APPROXIMATION

Shuey (1985) modified the Aki and Richards (2002) equation, replacing the properties v_S and Δv_S by σ and $\Delta\sigma$, obtaining the following result:

$$R_{PP}(\theta) \approx R_0 + \operatorname{sen}^2\theta\left[A_0 R_0 + \frac{\Delta\sigma}{(1-\sigma)^2}\right] + \frac{1}{2}(\operatorname{tg}^2\theta - \operatorname{sen}^2\theta)\left(\frac{\Delta v_P}{v_P}\right), \quad (11)$$

where

$$R_0 \approx \frac{1}{2}\left(\frac{\Delta v_P}{v_P} + \frac{\Delta\rho}{\rho}\right) \approx \frac{1}{2}\left(\frac{\Delta Z_P}{Z_P}\right),$$

$$A = A_0 + \frac{\Delta\sigma}{(1-\sigma)^2 R_0},$$

$$A_0 = B - 2(1+B)\frac{1-2\sigma}{1-\sigma},$$

$$B = \frac{\Delta v_P/v_P}{\Delta v_P/v_P + \Delta\rho/\rho}.$$

Equation (11) shows that combinations of elastic properties are effective in the angle θ range. The third term is proportional to θ^4, not normally contributing to angle values $\theta < 30^o$ (Barros, 1997). The dimensionless parameter A controls whether the amplitude initially increases ($A > 0$) or decreases ($A < 0$), while the dimensionless parameter B controls the signal for wide angles.

THOMSEN APPROXIMATION

Thomsen (1990) suggests the introduction of the rigidity module, μ, in Equation (11) instead of Poisson's ratio, as Shuey (1985) proposed in his work. According to Thomsen (1990), the Poisson's ratio is irrelevant for wave propagation, since it was defined in terms of a thin bar compression experiment. However, the relative magnitude of compressional and shear deformations is an important notion, but Poisson's ratio is not a good way to express this magnitude. The solution found by Thomsen (1990) was to write Poisson's ratio in terms of other parameters, such as the ratio between compressional and shear velocities, obtaining the following expression:

$$R_{PP}(\theta) \approx \frac{1}{2}\left(\frac{\Delta Z_P}{Z_P}\right) - 2\left(\frac{v_S}{v_P}\right)^2 \text{sen}^2\theta \frac{\Delta\mu}{\mu} + \frac{1}{2}\text{tg}^2\theta \left(\frac{\Delta v_P}{v_P}\right), \quad (12)$$

where

$$\frac{\Delta\mu}{\mu} = 2\frac{\Delta v_S}{v_S} + \frac{\Delta\rho}{\rho}.$$

VALIDITY OF THE APPROXIMATE EQUATIONS

The validity of the approximate equations was tested using the model studied by Ostrander (1984), shown in Table 1. Three pairs of Poisson's ratio values were used, associated with the different values of velocity and density. The first case considers that there is no variation in the Poisson's ratio between the layers. The values $\sigma_1 = \sigma_2 = 0.3$ were used. The second model considers that the Poisson's ratio of the upper layer is greater than that of the lower layer; the values $\sigma_1 = 0.4$ and $\sigma_2 = 0.1$ were used. Finally, the third case considers that the Poisson's ratio of the upper layer is less than that of the lower layer. The adopted values were $\sigma_1 = 0.1$ and $\sigma_2 = 0.4$. The approximate equations of the seismic reflection coefficient showed similar behavior in the three analyzed cases. In case 1, shown in Figure 1, we consider that the velocity ratio and also the density ratio are greater than 1; in this case, $v_{P2}/v_{P1} = \rho_2/\rho_1 = 1.25$. It can be seen that the errors were less than 10% for incidence angles up to 30°. In case 2, for velocity and density contrasts smaller than 1, for example, $v_{P2}/v_{P1} = \rho_2/\rho_1 = 0.8$, the approximate equations curves shown in Figure 2 present good accuracy for all the incidence angles considered. In case 3, shown in Figure 3, considering $v_{P2}/v_{P1} = \rho_2/\rho_1 = 1.25$, the approximate equations presented good accuracy for angles up to 30°.

Table 1 - Model proposed by Ostrander (1984) for different values of the Poisson ratio.

Case	σ_1	σ_2
1	0.3	0.3
2	0.4	0.1
3	0.1	0.4

ANALYSIS OF SENSITIVITY IN SYNTHETIC MODELS

The linearized expressions of the reflection coefficient can be written as a general expression:

$$R_{PP}(\theta) \approx g_{11}m_1 + g_{12}m_2 + g_{13}m_3, \quad (13)$$

where g_{1j} are the elements of the matrix $\boldsymbol{G}(\theta)$ and m_i are the model parameters to be estimated. In this case, the matrix $\boldsymbol{G}(\theta)$ can be represented by discrete values of θ:

$$\boldsymbol{G}(\theta_k) \approx [g_{11}(\theta_k)\ g_{12}(\theta_k) g_{13}(\theta_k)], \quad k \in \{0,1,2,\cdots k-1\}. \quad (14)$$

The matrix $\boldsymbol{G}^T(\theta_k)\boldsymbol{G}(\theta_k)$ is given by (Barros, 1997; Barros and Ramos, 1997):

$$\boldsymbol{G}^T(\theta_k)\boldsymbol{G}(\theta_k) = \begin{pmatrix} \sum g_{11}(\theta_k)^2 & \sum g_{11}(\theta_k)g_{12}(\theta_k) & \sum g_{11}(\theta_k)g_{13}(\theta_k) \\ \sum g_{11}(\theta_k)g_{12}(\theta_k) & \sum g_{12}(\theta_k)^2 & \sum g_{12}(\theta_k)g_{13}(\theta_k) \\ \sum g_{11}(\theta_k)g_{13}(\theta_k) & \sum g_{12}(\theta_k)g_{13}(\theta_k) & \sum g_{13}(\theta_k)^2 \end{pmatrix}. \quad (15)$$

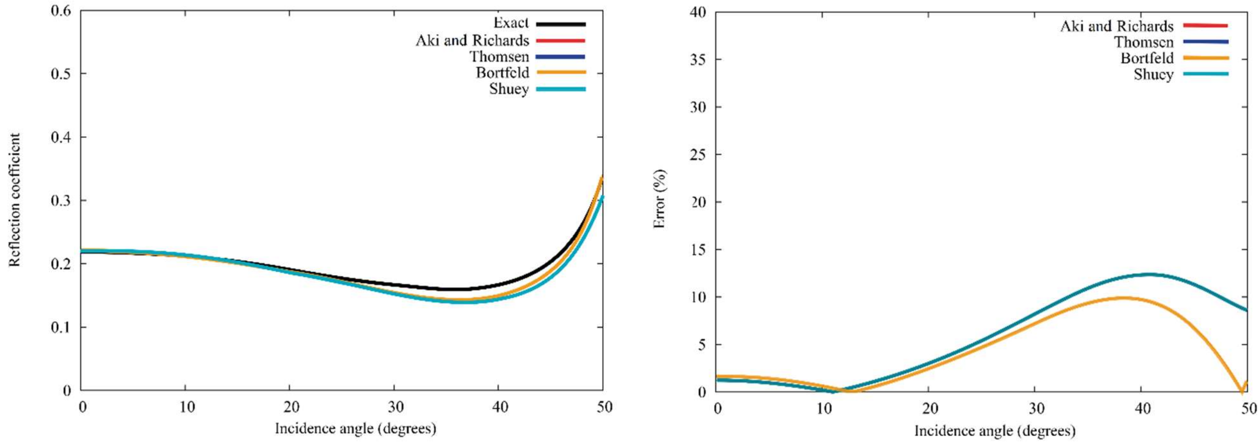

Figure 1 - Simulations with the Ostrander (1984) model, case 1, with $\sigma_1 = \sigma_2 = 0.3$ and $v_{P2}/v_{P1} = \rho_2/\rho_1 = 1.25$. The upper graph shows the reflection coefficient as a function of the incidence angle for the I and Richards, Thomsen, Bortfeld and Shuey approximations, in addition to the exact expression. The graph at the right shows the error for each approximation.

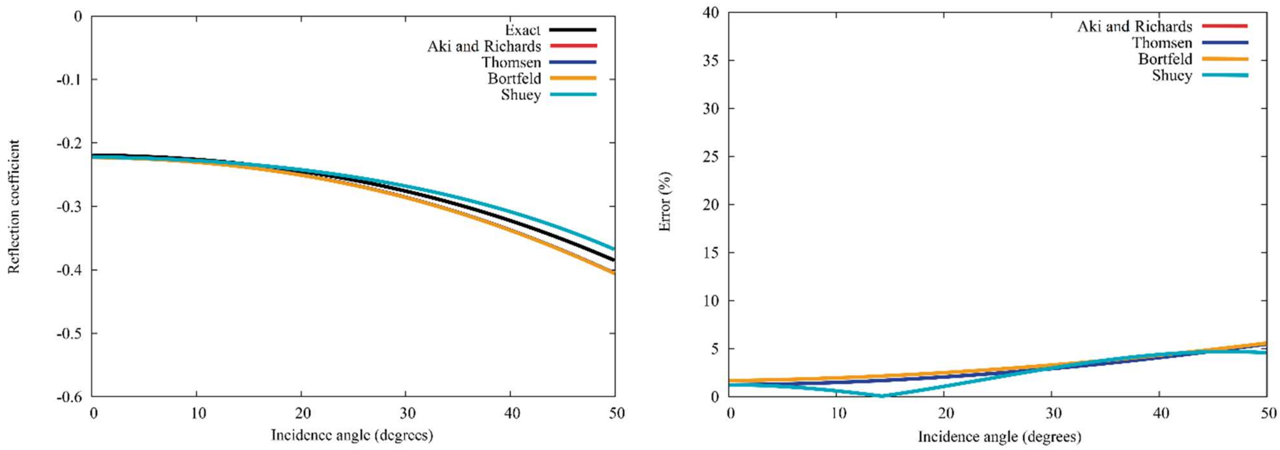

Figure 2 - Simulations with the Ostrander (1984) model, case 2, with $\sigma_1 = 0.4$, $\sigma_2 = 0.1$ and $v_{P2}/v_{P1} = \rho_2/\rho_1 = 0.8$. The upper graph shows the reflection coefficient as a function of the incidence angle for the Aki and Richards, Thomsen, Bortfeld and Shuey approximations, in addition to the exact expression. The graph at the right shows the error for each approximation.

We consider that the data space is restricted from the normal incidence ($\theta = 0^o$) to a maximum angle θ_{max} which is smaller than the critical angle. We will also consider that the data is sampled uniformly for incidence angles with a sampling interval $\Delta\theta$.

We study the sensitivity of the inversion process for the approximation of Aki and Richards (2002), presented in terms of the equation $\boldsymbol{d} = \boldsymbol{G}\boldsymbol{m}$, where the vector $\boldsymbol{d}$ with dimension k represents the reflection coefficient and is expressed as

$$\boldsymbol{d}^{obs}(\theta_k) = R_{pp}(\theta_k). \quad (16)$$

The matrix $\boldsymbol{G}$, with three columns and k rows, is expressed as

$$\boldsymbol{G}(\theta_k) = \left(\frac{1}{2\,cos^2\,\theta_k}, \frac{1}{2}\left[1 - 4\left(\frac{v_S}{v_P}\right)^2 \mathrm{sen}^2\theta_k\right], 4\left(\frac{v_S}{v_P}\right)^2 \mathrm{sen}^2\theta_k\right), \quad (17)$$

and the vector $\boldsymbol{m}$, with three parameters, is expressed as

$$\boldsymbol{m} = \left(\frac{\Delta v_P}{v_P}, \ \frac{\Delta\rho}{\rho}, \ \frac{\Delta v_S}{v_S}\right)^T. \quad (18)$$

Through the matrix $\boldsymbol{G}(\theta_k)$, the reflectivity $\boldsymbol{d}^{obs}(\theta_k)$ depends on the parameter v_P/v_S. In this work, the value $\sqrt{3}$ for this ratio will be used (Nicolao et al., 1993), which corresponds to $\sigma = 0.25$ and is called Poisson solid.

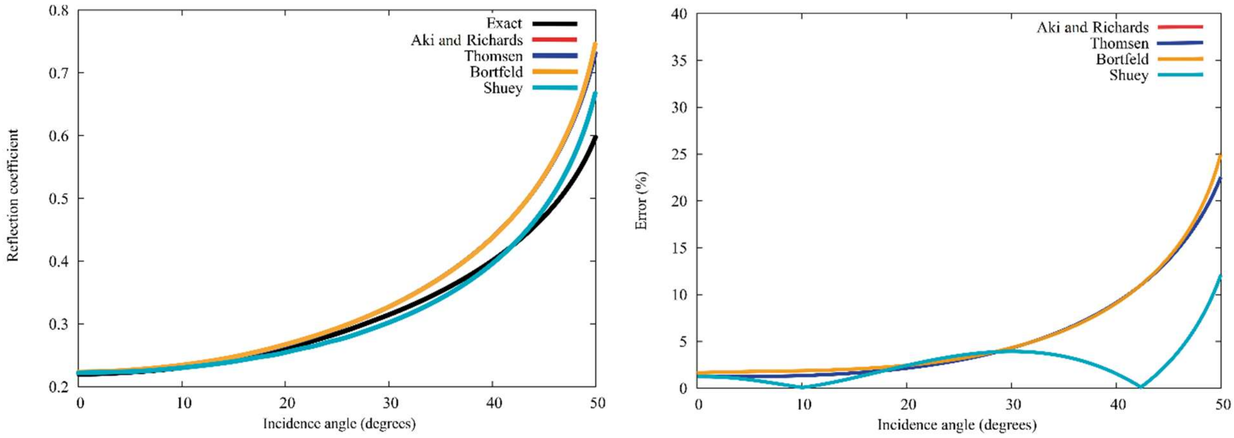

Figure 3 - Simulations with the Ostrander (1984) model, case 3, with $\sigma_1 = 0.1$, $\sigma_2 = 0.4$ and $v_{P2}/v_{P1} = \rho_2/\rho_1 = 1.25$. The upper graph shows the reflection coefficient as a function of the angle of incidence for the Aki and Richards, Thomsen, Bortfeld and Shuey approximations, in addition to the exact expression. The graph at the right shows the error for each approximation.

The three eigenvalues are functions of the maximum incidence angle and the unit of measurement is the logarithmic scale dB. 20 dB corresponds to a ratio of 10 in amplitude or 100 in energy. For the Aki and Richards (2002) approximation, the first eigenvalue (eigenvalue 1) contains almost all the signal energy, as can be seen in Figure 4. For a maximum incidence angle of 30°, the energy associated with the first eigenvalue it is 100 times greater than the energy associated with the second, and it is 10,000 times greater than the energy associated with the third one. The eigenvalue 2 is negligible for small incidence angles, but its value grows as the incidence angles increase. For large angles, eigenvalue 2 is from 10 to 15 dB smaller than eigenvalue 1. Eigenvalue 3 has the lowest energy, being easily masked by noise (Nicolao et al., 1993).

AUTOVECTORS IN MODEL SPACE

The analysis of eigenvectors in the model space aims to determine which parameters can be obtained with better precision in the inversion process. For this, we calculate the cosine directors of each eigenvector and analyze the direction in which each one points. Studying the eigenvector associated with the first eigenvalue in the Aki and Richards (2002) approximation, we see in Figure 5 that for small angles, the variation in the P-wave velocity (eigenvector 1) and the variation in density (eigenvector 2) are similar, while the variation in the velocity of the S wave (eigenvector 3) is close to zero.

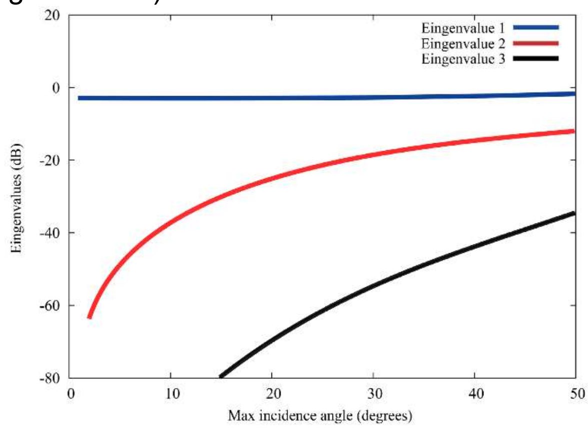

Figure 4 - Relative amplitude of the eigenvalues as a function of the maximum incidence angle, using Aki and Richards (2002) approximation.

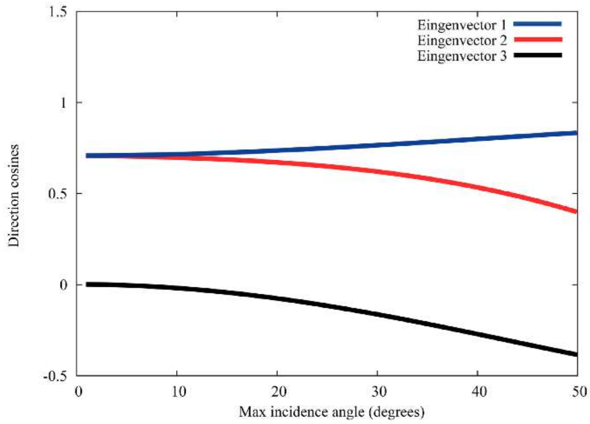

Figure 5 - Cosine directors of the eigenvectors corresponding to the first eigenvalue calculated by Aki and Richards (2002) approximation.

Therefore, the first vector points in the direction of the P-wave impedance variation, expressed by:

$$\frac{\Delta Z_P}{Z_P} = \frac{\Delta\rho}{\rho} + \frac{\Delta v_P}{v_P}. \tag{19}$$

The eigenvector corresponding to the second eigenvalue is illustrated in Figure 6 and points to the sum of the S-wave velocity variation and the density variation, thus obtaining the S-wave impedance:

$$\frac{\Delta Z_S}{Z_S} = \frac{\Delta\rho}{\rho} + \frac{\Delta v_S}{v_S}. \tag{20}$$

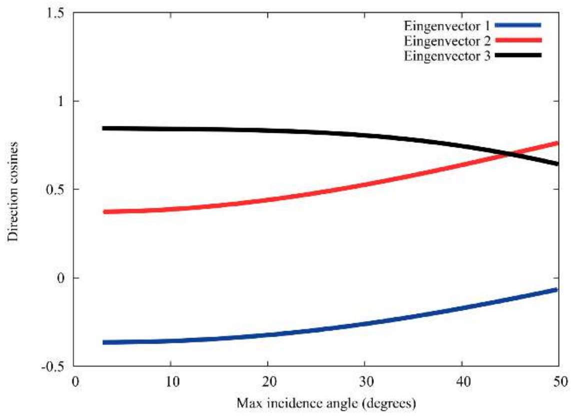

Figure 6 - Cosine directors of the eigenvectors corresponding to the second eigenvalue calculated by Aki and Richards (2002) approximation.

Finally, the eigenvector corresponding to the third eigenvalue, shown in Figure 7, has very low energy, indicating no particular physical property.

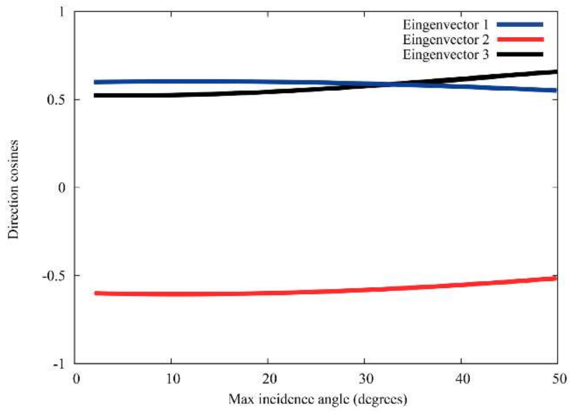

Figure 7 - Cosine directors of the eigenvectors corresponding to the third eigenvalue calculated by Aki and Richards (2002) approximation.

Based on these results, we can reformulate Equation (10) in terms of the P- and S-wave impedances, obtaining:

$$R_{PP}(\theta) \approx \frac{1}{2}(1 + tan^2\,\theta)\frac{\Delta Z_P}{Z_P} - 4\left(\frac{v_S}{v_P}\right)^2 \text{sen}^2\theta\frac{\Delta Z_S}{Z_S} - \left[\frac{1}{2}tan^2\,\theta - 2\left(\frac{v_S}{v_P}\right)^2 \text{sen}^2\theta\right]\frac{\Delta\rho}{\rho}. \tag{21}$$

Next, we compare the results of the inversion process and the sensitivity analysis. We call Equation (10) the first parameterization and Equation (21) the second one. This procedure is very important, as we see if, in fact, $\Delta Z_P/Z_P$ can be more precisely determined, compared with $\Delta v_P/v_P$ and $\Delta\rho/\rho$ separately.

The analysis of eigenvalues and eigenvectors in the model space provides valuable information on the parameters to be obtained in the inversion process. The eigenvalues revealed the great difficulty of obtaining the three parameters defined in each approximate equation. Figure 4 shows that only the first eigenvalue can provide some information for all the considered maximum incidence angles. The second eigenvalue is only relevant for wide angles and the third eigenvalue is negligible. The simultaneous use of the three eigenvalues in the inversion process can compromise the quality of the parameter estimation. According to Figures 5, 6 and 7, in the approximation of Aki and Richards (2002), the parameters are not well determined individually, being a combination of them more appropriate for the estimate, as shown in Equations (19) and (20). In this case, the best estimated physical properties would be the P- and S-wave impedances instead of $\Delta v_P/v_P$, $\Delta v_S/v_S$ and $\Delta\rho/\rho$.

Linear Inversion in Synthetic Models using Aki and Richards Approximation

Linear inversion aims to recover elastic parameters using approximate equations, which are the theoretical basis of AVO analysis. Each equation contains three explicit parameters that can be estimated, with the assumption that the reflection coefficient is known. We can estimate the model parameters using a least squares solution, however, the classic inverse of the matrix $\boldsymbol{G}^T\boldsymbol{G}$ can be replaced by the generalized inverse, which is obtained by the SVD technique. Thus, the elastic parameters can be estimated by linear inversion by (Aster et al., 2016):

$$\boldsymbol{m}^{est} = (\boldsymbol{G}^T\boldsymbol{G})^{+}\boldsymbol{G}^T\boldsymbol{d}^{obs}. \quad (22)$$

The observed data represent the reflection coefficient as a function of the incidence angle and the model parameters to be estimated represent the elastic parameters defined in the approximate equation.

It is important to note that the approximate equations are subject to limitations that allow the linearization of elastic modeling and elastic inversion. The limitation of small contrasts in the parameters simplifies the estimation of the elastic parameters, but causes a loss of information, since the approximation is limited to values of incidence angles smaller than the critical angle.

We used the Ostrander (1984) model for inversion, which consists of a gas-filled sandstone reservoir embedded in a shale pack. The tests were performed for two interfaces. The first interface corresponds to the top of the layer (shale/sandstone) and the second interface corresponds to the base of the layer (sandstone/shale). Table 2 shows the values in each layer of the P-wave velocity, the S-wave velocity, the density and the Poisson's ratio. The inversion was performed for a wide range of maximum incidence angles, from 0° to 50°. We analyzed how each parameter behaves in the considered interval.

Table 2 - Reservoir filled with gas for inversion of synthetic data. Values taken from Ostrander (1984).

Layer	Lithology	$v_P\left(\frac{m}{s}\right)$	$v_S\left(\frac{m}{s}\right)$	$\rho\left(\frac{g}{cm^3}\right)$	σ
1	Shale	3048.0	1244.3	2.40	0.40
2	Sandstone with gas	2438.4	1625.6	2.14	0.10
3	Shale	3048.0	1244.3	2.40	0.40

RESULTS AND DISCUSSIONS

The three parameters estimated in the first parameterization of the Aki and Richards (2002) approximation in Equation (9) are: P-wave reflectivity, $\Delta v_P/v_P$ (parameter 1), density variation, $\Delta\rho/\rho$ (parameter 2) and S-wave reflectivity, $\Delta v_S/v_S$ (parameter 3).

The result of the inversion of the top of the layer is shown in Figure 8. The estimate of the parameters using three eigenvalues is not satisfactory, but with the exclusion of the two smallest eigenvalues the result improves. As demonstrated in the sensitivity analysis, the first parameter can be well recovered, being better estimated for an angle range between 30° and 40°. The second and third parameters are difficult to recover, and a good estimate is not possible in the inversion process.

Figure 9 shows the result of the inversion of the base of the layer. It can be seen that the estimate of the first parameter for maximum incidence angles in the range of approximately 30° and 50° is very good, with less than 10% errors. The second parameter is only well estimated for an angle of around 45°, where the associated error is close to zero. The estimate of the third parameter is unsatisfactory with a high error.

The second parameterization of the Aki and Richards (2002) approximation, presented in equation (21), was also used. The three parameters to be obtained are: P-wave impedance variation, $\Delta Z_P/Z_P$ (parameter 1), S-wave impedance variation, $\Delta Z_S/Z_S$ (parameter 2) and relative density variation, $\Delta\rho/\rho$ (parameter 3).

Figure 10 shows the result of the inversion at the top of the layer. The first parameter was well obtained for all incidence angles, corroborating perfectly with the sensitivity analysis. The error increases for large incidence angles, being smaller than 10% for a maximum angle of 50°. The estimate of the first parameter for this parameterization is much better than the previous one, when equation (10) was used. However, obtaining the second and third parameters proved to be very inconsistent, with a very high error for the analyzed angle range.

Figure 11 shows the result of the inversion at the base of the layer with little difference from the top. The first parameter is still well estimated, however, limited to an angle range from 0° to 35°. The other parameters are not estimated satisfactorily, despite the fact that the errors are reduced for larger angles.

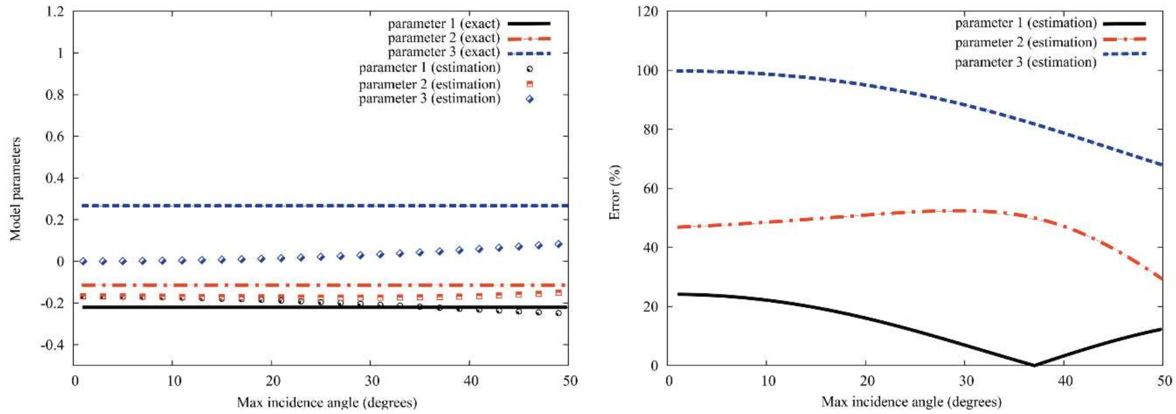

Figure 8 - Result of the inversion of synthetic data using the first parameterization of Aki and Richards (2002) approximation in the shale/sandstone interface, top of the layer. The graph at the right shows the error for each estimated parameter.

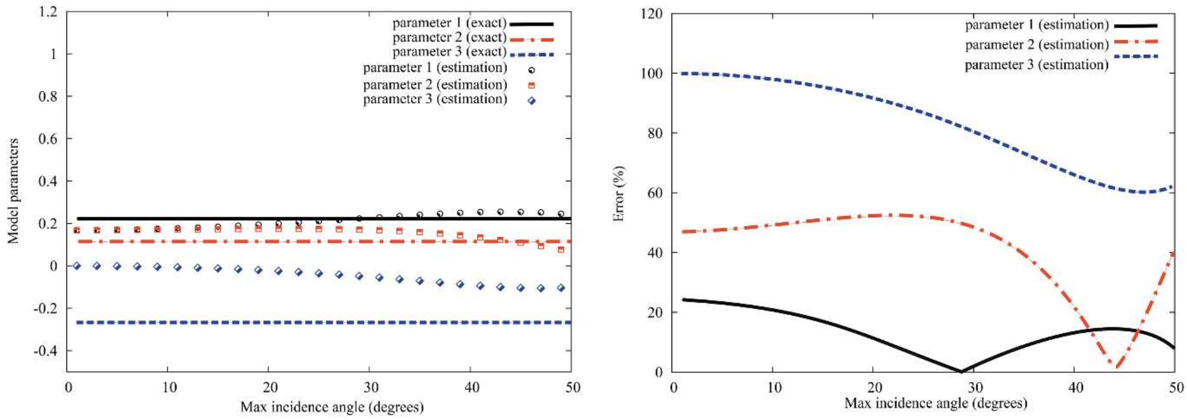

Figure 9 - Result of the inversion of synthetic data using the first parameterization of Aki and Richards (2002) approximation in the sandstone/shale interface, base of the layer. The graph at the right shows the error for each estimated parameter.

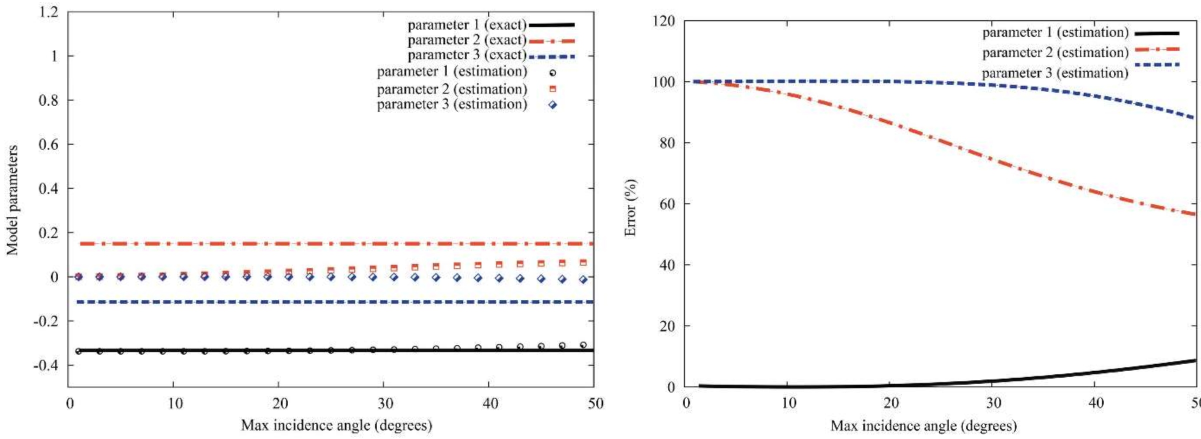

Figure 10 - Result of the inversion of synthetic data using the second parameterization of Aki and Richards (2002) approximation in the shale/sandstone interface, top of the layer. The graph at the right shows the error for each estimated parameter.

The sensitivity analysis of the inversion process of the linearized expressions showed the ill-conditioning of the problem, but the angle range from 0º to 50º showed that it is possible to accurately estimate the first two parameters. The third parameter could not be estimated accurately in either case. Tests with the Aki and Richards (2002) approximation showed that the second parameterization, presented in Equation (21), recovers the first parameter ($\Delta Z_P/Z_P$) very well, showing that the sum of the parameters $\Delta v_P/v_P$ and

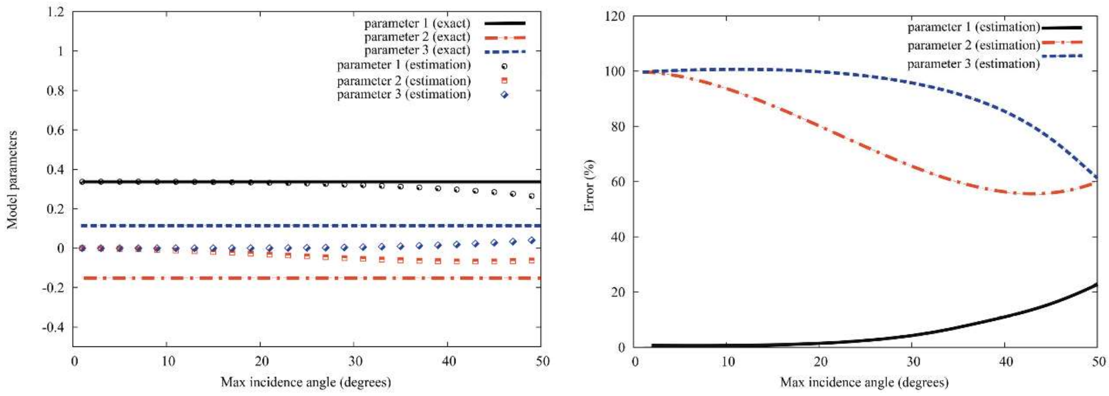

Figure 11 - Result of the inversion of synthetic data using the second parameterization of Aki and Richards (2002) approximation in the sandstone/shale interface, base of the layer. The graph at the right shows the error for each estimated parameter,

$\Delta\rho/\rho$ is better determined than each one separately. In the first parameterization, Equation (9), there is a possibility to satisfactorily determine the second parameter ($\Delta\rho/\rho$), but only at the base of the layer.

CONCLUSIONS

The approximations of the exact expression of the seismic reflection coefficient present an excellent precision, when we take into account small contrasts in the parameters and incidence angles below the critical angle. In all tests, the approximations showed an error of less than 10% for angles up to 30°, except for two models that had different Poisson's ratio. The natural tendency of curves in approximate equations is to move away from the exact curve as the incidence angle increases. We analyzed the eigenvalues and eigenvectors in the model space using the SVD technique. With this procedure we obtained the estimated parameters and the conditioning of the system. The eigenvalue analysis revealed the difficulty of obtaining the three parameters in the inversion process. We studied the behavior of each parameter for maximum incidence angles in a range of 0° to 50°. The inverse matrix was obtained by the SVD technique, truncating the two smallest eigenvalues. In all the studied approximations, the first parameter is the one with the lowest percentage error and, depending on the maximum incidence angle, the estimated value coincides with the exact value. The second parameterization of the Aki and Richards approximation corroborated perfectly with the sensitivity analysis, demonstrating that the P-wave velocity variation and the relative density variation can be better determined together, thus forming a impedance variation, rather than separately.

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D.L.S.: developed this study, processed the data, generated the figures, interpreted the results and wrote the first version of the manuscript. **A.B.:** supervised the research, contributed to the results discussion, and translated and reviewed the manuscript.

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