A study of the use of parameter constraints in an underdetermined inverse problem

Gabriel Pinto Santos (UFPA) and Cícero Régis (UFPA, INCT-GP)

Abstract

This study presents an example of 1D modeling and inversion of synthetic data from horizontal and vertical magnetic dipoles, focusing on configurations used in the EM34 equipment. The inversion code is an implementation of the Gauss-Newton method with Levenberg-Marquardt iterations, incorporating smoothness constraints on the model parameters. 1D inversion was conducted for three interpretative models: one with the same number of observations and parameters to be determined, and two others with fewer observations than parameters. As expected, the results indicate that in underdetermined models, a priori information helps to regularize the problem, reducing non-uniqueness and providing stable and reliable solutions.

Introduction

In this work, an illustration of 1D nonlinear inversion of data from horizontal and vertical magnetic dipoles, focusing on the EM34 equipment, was used to demonstrate the necessity of introducing a priori information in an underdetermined problem, i.e., one in which there are less observations than parameters to be determined.

The EM34 equipment (McNeill, 1980) operates based on the Slingram method, an active electromagnetic technique used to estimate the electrical conductivity of shallow conductive structures.

The inversion of geophysical data is almost always an ill-posed problem because it atempts to gather more information about the parameters than what is contained in the measured data. Therefore, the inclusion of *a priori* information about the parameters is necessary to assure unique and stable solutions.

This work illustrates this necessity by inverting synthetic 1D electromagnetic data generated in a horizontally stratified medium.

EM34 data are specially well suited to such demonstration because of their small number of observations gathered at each survey position.

The inversion code is an implementation of the Gauss-Newton method with Levenberg-Marquardt iterations (Gómez-Treviño et al., 2002; Pujol, 2007), incorporating smoothness constraints (Constable et al., 1987) on the model parameters, which in this case are the resistivities of the layers.

Methodology

The Slingram method used in the EM-34 equipment enables the estimation of electrical conductivity in the subsurface through measurements of secondary magnetic field variations (McNeill, 1980). This method operates in the frequency domain, measuring the electromagnetic field and its interactions with the lithological medium. The equipment consists of two coils: the first coil, energized by alternating current, generates the primary electromagnetic field, which induces currents in the subsurface, thereby producing a secondary electromagnetic field. The second coil, acting as a receiver, is positioned at fixed distances from the transmitter in predetermined configurations.

In the coplanar configurations used by the EM34 equipment, the horizontal ($\mathbf{H}_y$) and vertical ($\mathbf{H}_z$) components of the magnetic fields are generated, respectively, by horizontal magnetic dipole (HMD) and vertical magnetic dipole (VMD) sources.

For the stratified problem (1D), determining the fields at the receiver position involves the numerical evaluation of improper integrals of the Hankel transform (Ward & Hohmann, 1988, pp. 208 and 223). These 1D solutions are used to compute the magnetic field in both forward and inverse problems. Digital filters, as presented by Werthmüller et al. (2019), are utilized for VMD integrals, while the Quadrature with Extrapolation method (Key, 2012) is employed for HMD integrals.

1D Nonlinear Inversion: Gauss-Newton Method with Marquardt Iterations

Given the field observations contained in the vector $\mathbf{y}$, the objective of the inversion is to determine the values of a parameter vector $\mathbf{p}$ that generate synthetic data best fitting the observations. In this case, the observed data are the apparent conductivities or the imaginary component of the magnetic field, and the model parameters are the resistivities of the layers.

The functional to be minimized $\Phi(\mathbf{p})$ is defined as:

$$\Phi(\mathbf{p}) = \phi^d(\mathbf{p}) + \alpha\, \phi^v(\mathbf{p}), \tag{1}$$

where

$$\phi^d(\mathbf{p}) = \frac{1}{2}\sum_{i=1}^{m}[\mathbf{y}_i^0 - f_i(x_i, \mathbf{p})]^2 \tag{2}$$

is the functional that represents the fitting of observations

by the generated data from the mathematical model f, which depends on **p** and the independent variables such as frequency and measurement positions. The constraint function

$$\phi^v(\mathbf{p}) = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2)^2 + (\mathbf{p}_2 - \mathbf{p}_3)^2 + \cdots + (\mathbf{p}_{n-1} - \mathbf{p}_n)^2] \quad (3)$$

defines *a priori* equality relationships between adjacent parameters to enforce smoothness in the solution. In the objective function (1), α is a positive scalar known as the regularization parameter, whose function is to weigh the relative importance of the information provided by the constraint relationships:

The Gauss-Newton method defines the sensitivity matrix **A**, whose elements

$$\mathbf{A} = \frac{\partial \phi_i^d}{\partial p_j} = -\sum_{i=1}^{m} [\mathbf{y}_i^0 - f_i(x_i, p_j)] \frac{\partial \phi_i^d}{\partial p_j}, \quad (4)$$

and the residual vector

$$\delta \mathbf{y}_i = \mathbf{y}_i^0 - f_i(x_i, \mathbf{p}_j). \quad (5)$$

Then the method produces an estimate of **p** that minimizes the objective function (eq. 1) by iterating the equation

$$(\mathbf{A}^T\mathbf{A} + \alpha\mathbf{M}^T\mathbf{M})\delta\mathbf{p}_k = \mathbf{A}^T\delta\mathbf{y} + \alpha\mathbf{M}^T\mathbf{M}\mathbf{p}. \quad (6)$$

The relationships between the parameters defining the constraint function (eq. 3) are expressed in the matrix **M** as

$$\mathbf{M} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad (7)$$

which leads to the square matrix

$$\mathbf{M}^T\mathbf{M} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \quad (8)$$

The Marquardt technique aims to ensure that the convergence estimate always takes one step in the gradient descent direction. This is achieved by alternating between an estimate using the Gauss-Newton method and taking a small step in the gradient descent direction. Combining the two types of steps to minimize the fit functional into a single discovery is done by adding a positive scalar λ, called the Marquardt parameter, to the diagonal of the Hessian matrix of the functionals, so that the iteration becomes

$$(\mathbf{A}^T\mathbf{A} + \alpha\mathbf{M}^T\mathbf{M} + \lambda I), \delta\mathbf{p}_k = \mathbf{A}^T\delta\mathbf{y} + \alpha\mathbf{M}^T\mathbf{M}\mathbf{p}, \quad (9)$$

The value of λ is decreased by one iteration if the value of the objective function Φ decreases. Otherwise, its value is increased and a new $\delta\mathbf{p}$ is generated. This step is repeated until Φ decreases in the iteration.

The iteration of the equation 9 is repeated until the objective function reaches a minimum or a maximum number of iterations is reached, which means that the process has not converged.

Results and discussions

The model that generated the synthetic data is composed of only three layers including an infinite basement (Figure 1). The first layer has a resistivity of 50 Ω m and a thickness of 10 m, the second layer is more conductive and has a resistivity of 10 Ω m with a thickness of 20 m, and the basement has a resistivity of 100 Ω m with infinite thickness.

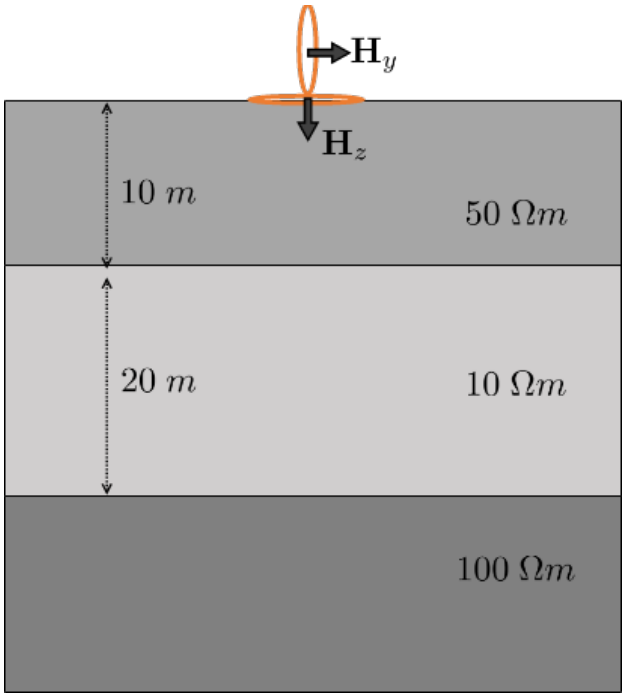

Figure 1: 1D model that generated the synthetic data.

The 1D inversion was performed simulating the three configurations of the EM34 equipment, with three measurements for each of the DMH and DMV coplanar configurations, defined by three offset/frequency pairs: 10 m at 6.4 kHz; 20 m at 1.6 kHz; and 40 m at 400 Hz. The regularization parameter $\alpha = 10^{-6}$, 10^{-9} and 10^{-8}. Three interpretative models were used. The first model consists of three layers, the last being the basement, all starting with a resistivity of 10 $\Omega \cdot m$. The first layer is 10 m thick, and the second layer is 25 m thick. The second model has twice as many layers as the first, with thicknesses increasing by 1 m with depth up to the penultimate layer. The first layer is 6 m thick and the last layer is 15 m thick, all with resistivities of 10 $\Omega \cdot m$. In the third model, composed of 13 layers, all with

resistivity of 10 $\Omega \cdot m$. The thickness of the layers increases with depth by 1.5 m, starting with the first layer which is 2 m thick.

Even in systems where the number of observations is equal to the number of parameters, the uniqueness and stability of the solution are not guaranteed. This behavior can be seen in figure 3, where the smoothness constraint has not been applied. Although a well-determined system suggests the existence of a unique solution, in practice these problems can be ill-conditioned. This makes it necessary to introduce a priori information to stabilize the solution, mitigating uncertainties in the observation data, as shown in figure 2.

The inclusion of the smoothness constraints imposes restrictions on the spatial variations of the estimated parameters, promoting more plausible and geologically realistic models. To illustrate this behavior, figures 4 and 6 show that by introducing a priori information, the inversion is able to better estimate the resitivities of the medium.

In figures 5 and 7, where there is no a priori information, the estimates are unrealistic.

Conclusion

The introduction of the smoothness constraints in geophysical inversion not only improves the accuracy of model parameter estimates, but also strengthens reliability and contributes to better interpretation of the results.

Acknowledgements

To Petrobras for supporting this research through project number 2023/00329-4.

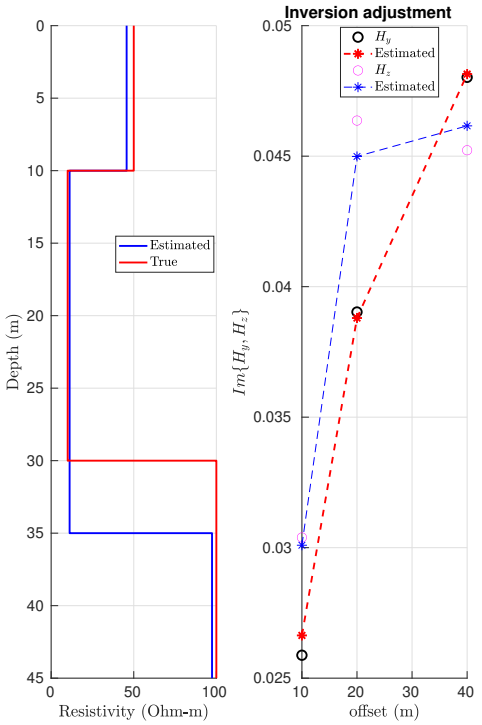

Figure 2: Result of the inversion with smoothness constraint for the 3-layer interpretive model, $\alpha = 1 \times 10^{-6}$.

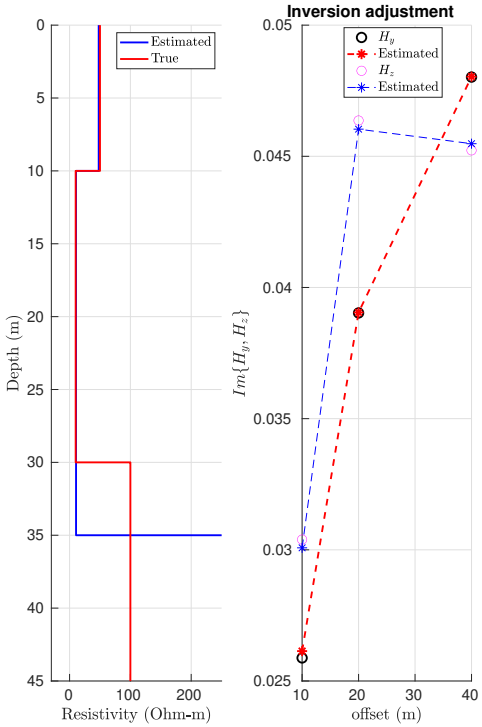

Figure 3: Result of the inversion without the smoothness constraint for the 3-layer interpretive model.

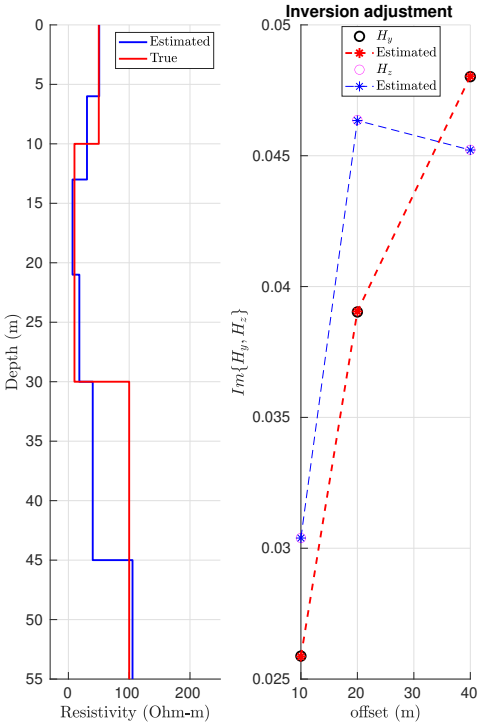

Figure 4: Inversion result with smoothness constraint for the 6-layer interpretive model, $\alpha = 1 \times 10^{-9}$.

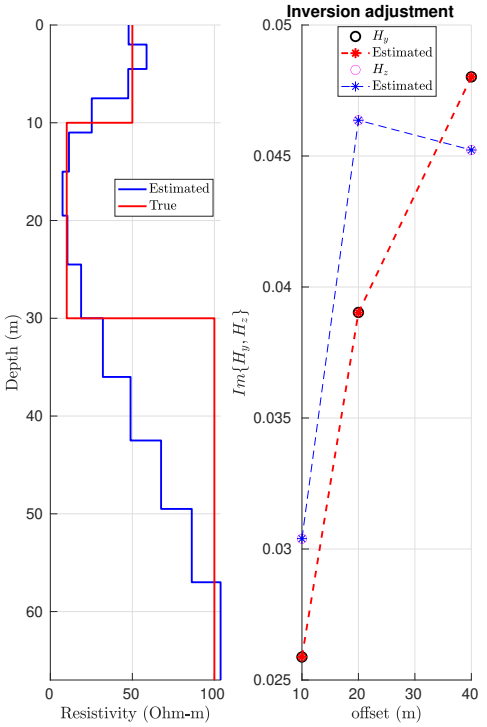

Figure 6: Result of the inversion with smoothness constraint for the 13-layer interpretive model, $\alpha = 1 \times 10^{-9}$.

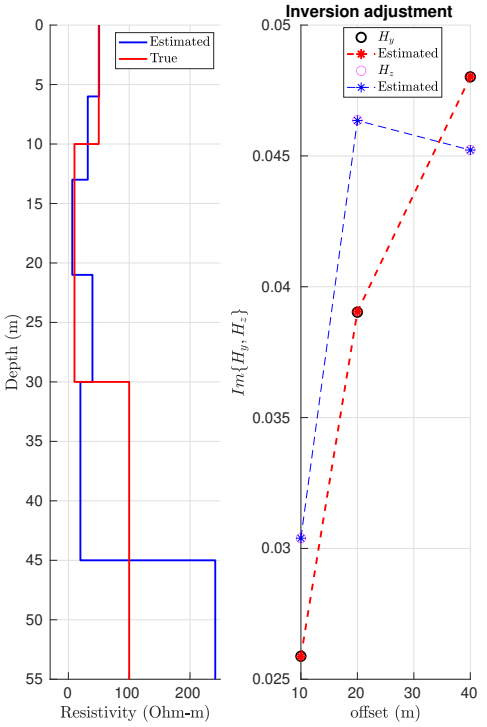

Figure 5: Result of the inversion without smoothness constraint for the 6-layer interpretive model.

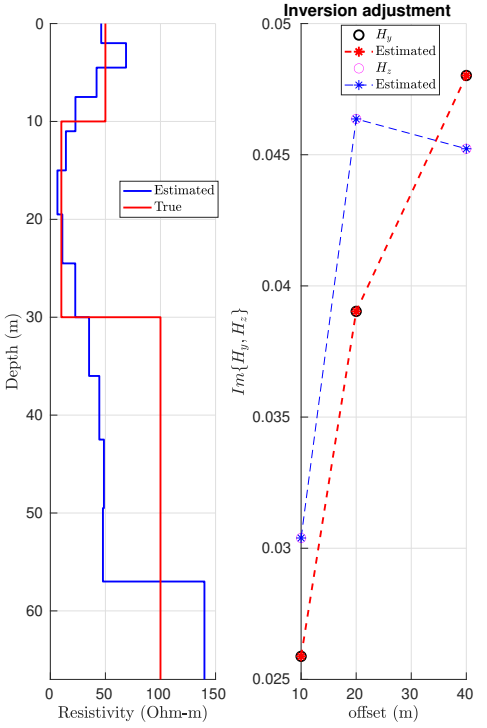

Figure 7: Result of the inversion without the smoothness constraint for the 13-layer interpretive model.

References

Constable, S. C., Parker, R. L. & Constable, C. G., 1987. Occam's inversion: a practical algorithm for generating smooth models from em sounding data, Geophysics, vol. 52(3): 289–300.

Gómez-Treviño, E., Esparza, F. J. & Méndez-Delgado, S., 2002. New theoretical and practical aspects of electromagnetic soundings at low induction numbers, Geophysics, vol. 67(5): 1441–1451.

Key, K., 2012. Is the fast hankel transform faster than quadrature?, Geophysics, vol. 77(3): F21–F30.

McNeill, J. D., 1980. Electromagnetic terrain conductivity measurements at low induction numbers, Technical Note TN-6, Geonics Ltd.

Pujol, J., 2007. The solution of nonlinear inverse problems and the Levenberg-Marquardt method, Geophysics, vol. 72(4): W1–W16, doi:10.1190/1.2732552.

Ward, S. H. & Hohmann, G. W., 1988. Electromagnetic theory for geophysical applications, in: Electromagnetic Methods in Applied Geophysics: Voume 1, Theory, Society of Exploration Geophysicists, 130–311.

Werthmüller, D., Key, K. & Slob, E. C., 2019. A tool for designing digital filters for the hankel and fourier transforms in potential, diffusive, and wavefield modeling, GEOPHYSICS, vol. 84(2): F47–F56, doi:10.1190/geo2018-0069.1.